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PROCESSING(U) SYSTEMS CONTROL TECHNOLOGY INC PALO ALTO
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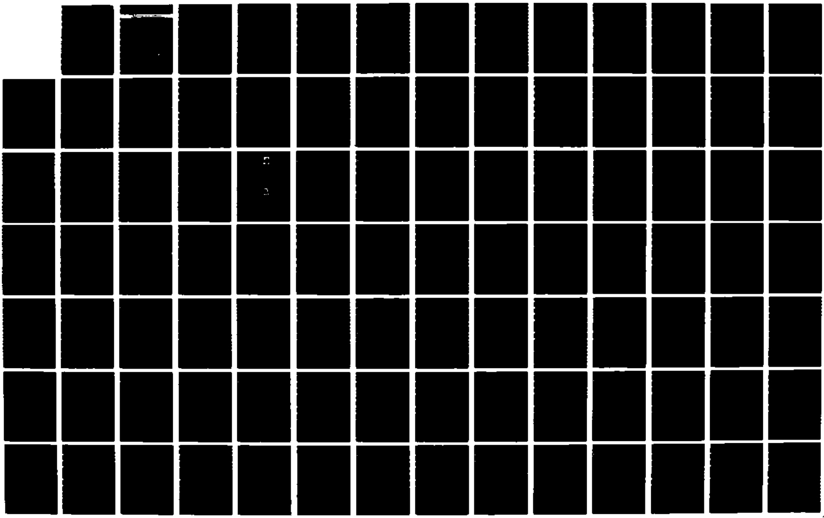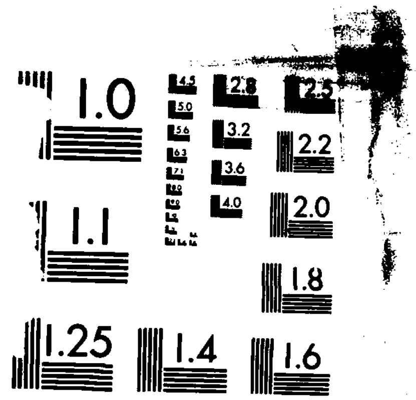

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PARAMETRIC TECHNIQUES FOR MULTICHANNEL SIGNAL PROCESSING

FINAL REPORT 5498-07

PREPARED BY:

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OCTOBER 1985

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CONTRACT NO. DAAG29-83-C-0027

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REPORT DOCUMENTATION PAGE

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1. REPORT NUMBER: 5498-07	2. GOVT ACCESSION NO.
3. RECIPIENT'S CATALOG NUMBER	
4. TITLE (and Subtitle): PARAMETRIC TECHNIQUES FOR MULTICHANNEL SIGNAL PROCESSING	5. TYPE OF REPORT & PERIOD COVERED: Final Report 9/1/83 - 10/30/85
	6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s): Benjamin Friedlander	8. CONTRACT OR GRANT NUMBER(s): DAAG29-83-C-0027
9. PERFORMING ORGANIZATION NAME AND ADDRESS: Systems Control Technology, Inc. 1801 Page Mill Road Palo Alto, CA 94304	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS: U.S. Army Research Office Post Office Box 12211 Research Triangle Park, NC 27709	12. REPORT DATE: October 1985
	13. NUMBER OF PAGES: 375
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office): U.S. Army Research Office Post Office Box 12211 Research Triangle Par k, NC 27709	15. SECURITY CLASS. (of this report): UNCLASSIFIED
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

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18. SUPPLEMENTARY NOTES

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19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Multichannel, Signal Processing, ARMA, Spectral Estimation, Yule-Walker, Adaptive

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This final report summarizes results on the development of parametric techniques for multichannel signal processing. Results include the development of accurate ARMA spectrum estimation techniques with modest computational complexity, and the development of asymptotic performance bounds for the evaluation of the accuracy of these techniques.

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Section

1. INTRODUCTION

This report briefly reviews the work performed under Army Research Office contract No. DAAG29-83-C-0027 on the development of parametric techniques for multichannel signal processing. The results are summarized in a number of papers, which are enclosed as appendices A-M.

1.1 MULTICHANNEL SIGNAL PROCESSING

Most of the work in the area of signal processing (in particular adaptive signal processing) is concerned with the single channel case: the design and analysis of filters with a single input and a single output (SISO). This type of processing is naturally suited to situations involving a scalar time series such as the video signal in a radar system or the output of a communication receiver. Many problems of great practical interest involve vector time series such as the signals in an acoustic or seismic array. To perform optimal prediction/estimation of such signals will usually require multi-input multi-output (MIMO) filters. Because of the higher complexity (both conceptual and computational) of MIMO filters, they are often replaced by suboptimal single channel processors.

In some recent work we developed a multichannel processor for the problem of estimating the parameters (location and spectrum) of multiple targets from multi-sensor data [10]. Preliminary simulation results indicated that significant performance improvements are achievable by performing optimal multichannel processing instead of the more conventional single channel processing. These initial positive results motivated us to study further the design and analysis of MIMO filters and their applications.

The advent of powerful VLSI processors makes it feasible to consider the more complex MIMO signal processing archietectures. The theoretical framework necessary for the development of multichannel processing techniques is currently available; researchers in system theory and modern control have been treating MIMO problems for the past two decades. We feel, therefore, that the time is right for the development and application of optimal multichannel signal processing techniques.

1.2 THE PARAMETRIC APPROACH

Autoregressive moving-average (ARMA) models are widely used in the statistical analysis of time series. In signal processing, autoregressive (AR) techniques have been used for high resolution spectral estimation, linear predictive coding, and (implicitly) in various adaptive filtering applications [1]-[3]. The use of ARMA models and the related infinite impulse response (IIR) prediction filters has been much more limited due to the difficulty of reliably estimating the parameters of such models from noisy data. Practical applications of these techniques have been limited to the single channel case.

In some recent work we applied (scalar) ARMA modeling techniques borrowed from the area of system identification to signal processing problems such as adaptive line enhancement, adaptive noise cancelling, adaptive deconvolution, and spectral estimation [4]-[7]. We also developed a very robust non-adaptive ARMA parameter estimation technique which was used for high resolution spectral estimation [11]. Other ARMA spectral estimation techniques were reported in [8]-[9]. Based on the accumulated experience with AR and ARMA signal processing techniques it seems that the single channel case is reasonably well developed by now. (It should be noted, however, that many questions are still open in the area of ARMA modeling for adaptive IIR filtering.)

The natural next step is to extend techniques for ARMA modeling to the MIMO case and to use them for designing multichannel signal processors. The main thrust of our research was, therefore, the development of robust estimation techniques for MIMO ARMA parameters. Once these parameters are estimated, they can be used to design MIMO filters for a variety of applications, as was shown in [4]-[7] for the single channel case. The problem of estimating MIMO ARMA parameters involves difficulties which were not present in the SISO case. These difficulties are related to the complex structure of MIMO systems and to questions of non-uniqueness of the representation of vector time-series.

2. PROJECT PUBLICATIONS

The following is a list of publications summarizing the work performed on this project. The key publications are included as appendices to this report.

In this project we developed a number of accurate ARMA estimation techniques which can be used for single and multichannel problems. These techniques require a modest amount of computation compared to a full-blown maximum likelihood technique. We have also developed asymptotic performance bounds that make it possible to evaluate the accuracy of these techniques. The results of this work are summarized in more than 30 project publications (see Section 2) and the key results are included in this report in appendices A-M. These results have a wide range of applications in the area of surveillance, communications, and statistical signal processing.

2.1 PUBLISHED JOURNAL PAPERS

1. B. Friedlander and B. Porat, "Some Bounds for the Estimation of Autoregressive Signals in White NNoise," Signal Processing, No. 8, pp. 291-302, 1985.

2. P. Stoica, T. Söderström and B. Friedlander, "Optimal Instrumental Variable Estimates of the AR Parameters of an ARMA Process," IEEE Trans. Automatic Control, Vol. AC-30, No. 11, pp. 1065-1075, November 1985.

2.2 ACCEPTED FOR PUBLICATION IN JOURNALS

3. P. Stoica, B. Friedlander and T. Söderström , "Least-Squares, Yule-Walker and Overdetermined Yule-Walker Estimation of AR Parameters: A Monte Carlo Study of Finite Sample Properties," Int. J. Control, to appear.

4. B. Porat and B. Friedlander, "Computation of the Exact Information Matrix for Gaussian Time Series with Stationary Random Components," IEEE Trans. Acoustics, Speech and Signal Processing, to appear.

5. B. Friedlander and B. Porat, "Multichannel Spectral Analysis Using the

Modified Yule-Walker Equations," J. Signal Processing, Special Issue on Spectral Estimation, to appear.

2.3 UNDER REVIEW

6. P. Stoica, B. Friedlander and T. Söderström, "Optimal Instrumental Variable Multistep Algorithms for the Estimation of AR Parameters of an ARMA Process."

7. P. Stoica, B. Friedlander and T. Söderström, "An Approximate Maximum Likelihood Estimator of ARMA Parameters."

8. P. Stoica, B. Friedlander and T. Söderström, "Maximum Likelihood Estimation of the Parameters of Multiple Sinusoids in Noise."

9. B. Porat and B. Friedlander, "Adaptive Detection of Deterministic Transient Signals."

10. B. Porat and B. Friedlander, "Asymptotic Performance Analysis of ARMA Parameter Estimation Methods Based on Sample Covariances," IEEE Trans. Automatic Control.

11. B. Porat and B. Friedlander, "The Exact Cramer-Rao Bound for Gaussian Autoregressive Processes," IEEE Trans. Information Theory.

12. P. Stoica, B. Friedlander and T. Söderström, "On Instrumental Variable Estimation of Sinusoid Frequencies and the Parsimony Principle," IEEE Trans. Acoustics, Speech and Signal Processing.

13. B. Friedlander, P. Stoica and T. Söderström, "Instrumental Variable Methods for ARMA Models," Chapter in Vol. XXIV of "Advances in Control and Dynamic Systems."

14. B. Porat and B. Friedlander, "Parameter Estimation of Continuous-Time Stationary Gaussian Processes with Rational Spectra," IEEE Trans. Acoustics, Speech and Signal Processing.

2.4 CONFERENCE PAPERS

15. B. Friedlander and B. Porat, "Multichannel Spectral Analysis Using the Modified Yule-Walker Equations," 18th ASILOMAR Conference on Circuits Systems and Computers, Pacific Grove, California, November 1984.

16. B. Friedlander and B. Porat, "Bounds for ARMA Spectral Analysis Based on Sample Covariances," Intl' Conf. Acoustics Speech and Signal Processing, Tampa, Florida, March 1985.

17. B. Porat and B. Friedlander, "Adaptive Detection of Transient Signals," Intl' Conf. Acoustics Speech and Signal Processing, Tampa, Florida, March 1985.

18. B. Porat and B. Friedlander, "Parameter Estimation of Continuous-time Stationary Gaussian Processes with Rational Spectra," Automatic Control Conference, Boston, Massachussetts, June 1985.

19. B.Porat and B. Friedlander, "Asymptotic Accuracy of ARMA Parameter Estimation Methods Based on Sample Covariances," 7th IFAC Symposium on Identification and System Parameter Estimation, York, United Kingdom, July 1985.

20. B. Friedlander, P. Stoica and T. Söderström , "Instrumental Variable Methods for ARMA Parameter Estimation," 7th IFAC Symposium on Identification and Sysstem Parameter Estimation, July 1985, York, United Kingdom.

21. B. Porat and B. Friedlander, "The Exact Cramer-Rao Bound for Gaussian Autoregressive Processes," 1985 Asilomar Conference on Circuits, Systems and Computers, Pacific Grove, California, November 6-8, 1985.

22. P. Stoica, B. Friedlander and T. Söderström , "Optimal Instrumental Variable Multistep Algorithms for Estimation of AR Parameters of an ARMA Process," The 1985 Conference on Decision and Control, Fort Lauderdale, Florida, December 1985.

23. P. Stoica, B. Friedlander and T. Söderström, "An Approximate Maximum Likelihood Estimator of ARMA Parameters," The 1985 Conference on Decision and Control, Fort Lauderdale, Florida, December 1985.

24. B. Porat and B. Friedlander, "Computation of the Exact Information Matrix for Gaussian Time Series with Stationary Random Components," The 1985 Conference on Decision and Control, Fort Lauderdale, Florida, December 1985.

2.5 REPORTS

25. B. Friedlander, "Parametric Techniques for Multichannel Signal Processing," Semi-Annual Progress Report 5498-01, April 1984.

26. P. Stoica, B. Friedlander and T. Söderström, "Maximum Likelihood Estimation of the Parameters of Multiple Sinusoids in Noise," Report 5498-02.

27. P. Stoica, B. Friedlander and T. Söderström, "An Approximate Maximum Likelihood Estimator of ARMA Parameters," Report 5498-03, December 1984.

28. P. Stoica, B. Friedlander and T. Söderström, "Optimal Instrumental Variable Multistep Algorithms for the Estimation of AR Parameters of an ARMA Process," Report 5498-04, December 1984.

29. P. Stoica, T. Söderström and B. Friedlander, "Optimal Instrumental Variable Estimates of the AR Parameters of an ARMA Process," Technical Report 54985-05, September 1984.

30. B. Porat and B. Friedlander, "Adaptive Detection of Deterministic Transient Signals," Technical Report 5498-05, September 1984.

APPENDIX A

ON THE COMPUTATION OF AN ASYMPTOTIC BOUND FOR ESTIMATING AUTOREGRESSIVE SIGNALS IN NOISE

Signal Processing 8 (1985) 291-302
North-Holland

ON THE COMPUTATION OF AN ASYMPTOTIC BOUND FOR ESTIMATING AUTOREGRESSIVE SIGNALS IN WHITE NOISE*

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Received 4 October 1984

Abstract. The Cramer-Rao lower bound (CRLB) provides a useful reference for evaluating the performance of parameter estimation techniques. This paper considers the problem of estimating the parameters of an autoregressive signal corrupted by white noise. An explicit formula is derived for computing the asymptotic CRLB for the signal and noise parameters. Formulas for the asymptotic CRLB for functions of the signal and noise parameters are also presented. In particular, the center frequency, bandwidth and power of a second order process are considered. Some numerical examples are presented to illustrate the usefulness of these bounds in studying estimation accuracy.

Zusammenfassung. Die Cramer-Rao untere Grenze (CRLB) gibt eine nützliche Referenz für die Performanz Evaluation von Parameter Estimationstechniken. Diese Kommunikation betrachtet das Problem der Estimation von Parametern eines autoregressiven Signals in weissem Rauchen. Eine explizite Formel wird angegeben für den asymptotischen CRLB von Signal und Rausch Parametern. Formeln für den CRLB von Funktionen der Signal und Rausch Parameter werden auch angegeben. In speziellen werden die Mittenfrequenz, Bandbreite und Leistung eines Prozesses zweiter Ordnung angegeben. Numerische Beispiele werden gegeben um die Nützlichkeit dieser Grenzen zu zeigen wenn Estimationsgenauigkeit studiert wird.

Résumé. La borne de Cramér Rao est un moyen utile pour évaluer l'efficacité d'une méthode d'estimation. On étudie dans cet article l'estimation de modèles de type AR plus bruit. On donne des formules explicites permettant de calculer de manière numeriquement efficace ces bornes de Cramer Rao. On examine de plus près le cas de la fréquence centrale et la bande passante d'une fréquence noyé dans du bruit. Ces résultats sont illustrés par des simulations.

Keywords. Autoregressive, Cramér Rao bound, asymptotic error analysis.

1. Introduction

The problem of estimating the parameters of signals from their noisy measurements arises in many engineering applications. A common model for a wide-sense stationary random signal is the autoregressive (AR) model. The signal is assumed to be corrupted by white measurement noise. In other words,

$$y_t = x_t + w_t, \tag{1}$$

where x_t is the signal, w_t is a zero-mean white noise process with variance σ_w^2, and y_t is the observed data. The autoregressive signal obeys the stochastic difference equation,

$$x_t = -\sum_{i=1}^{n} a_i x_{t-i} + u_t, \tag{2}$$

where u_t is a zero-mean white noise process with variance σ_u^2.

* This work was supported by the Army Research Office under Contract No. DAAG29-83-C-0027.

A considerable number of papers in the engineering and statistical literature treat the processing and estimation of autoregressive signals, see e.g., [1–6]. A useful tool for evaluating the performance of such AR estimation techniques, is the Cramer–Rao lower bound (CRLB) on the covariance matrix of the estimated parameters [7], [8]. Comparison of the covariance matrix of a given parameter estimation technique to the CRLB provides a measure of the accuracy of that technique.

While the CRLB has been known for quite some time, relatively little work seems to have been done on its computational aspects. In [4], [9] a simple numerical integration procedure for computing the CRLB is proposed, based on an asymptotic CRLB formula due to Whittle [10]. In the case of narrowband AR Processes considerable care must be taken to avoid excessive numerical errors. In this paper (Section 2) we present an explicit formula for the asymptotic CRLB for AR plus-noise processes, which does not involve numerical integration.

In many applications one is interested not in the AR parameters, but in some function of these parameters such as the center frequency, bandwidth and power of narrowband spectral lines. In Section 3 we present formulas for computing the CRLB of a general function of the AR-plus-noise parameters and of some special commonly used functions.

In Section 4 we present a few examples illustrating how to use the CRLB to study the effect of various signal and noise parameters on estimation accuracy.

2. An explicit formula for the Fisher information matrix

In this section we derive an explicit expression for the Fisher information matrix for the parameters $\{a_1, \ldots, a_n, \sigma_u^2, \sigma_w^2\}$. The inverse of the Fisher information matrix provides the Cramer–Rao lower bound on the estimation errors associated with these parameters. The derivation is somewhat lengthy, and will be performed in three steps. We start by introducing the spectral density function $S(z)$ of the AR-plus-noise process and computing its derivatives with respect to the various parameters. Using Whittle's formula for the asymptotic form of the Fisher information matrix [10] we express the entries of this matrix by complex integrals involving the spectrum $S(z)$ and its derivatives. Finally we evaluate these complex integrals using certain facts from the theory of discrete Lyapunov equations.

2.1. The spectrum and its derivatives

The spectrum of an AR-plus-Noise process defined in (1) (2) is given by

$$S(z) = \frac{\sigma_u^2}{a(z)a(z^{-1})} + \sigma_w^2 = \frac{\sigma_u^2 + \sigma_w^2 a(z)a(z^{-1})}{a(z)a(z^{-1})}, \tag{3}$$

where

$$a(z) = 1 + a_1 z + \cdots + a_n z^n, \tag{4}$$

has all of its roots outside the unit circle. Let $c(z)$ and K be defined by

$$\sigma_u^2 + \sigma_w^2 a(z)a(z^{-1}) = Kc(z)c(z^{-1}), \tag{5}$$

where $c(z)$ is the unique monic stable spectral factor of the left-hand side of (5), i.e.,

$$c(z) = 1 + c_1 z + \cdots + c_n z^n, \tag{6}$$

and $c(z)$ has all of its roots outside the unit circle. $S(z)$ and its inverse are given by

$$S(z) = \frac{Kc(z)c(z^{-1})}{a(z)a(z^{-1})};$$

$$S^{-1}(z) = \frac{a(z)a(z^{-1})}{Kc(z)c(z^{-1})}. \tag{7}$$

To compute the Fisher information matrix we need expressions for the partial derivatives of $S(z)$ with respect to the parameters $\{a_1, \ldots, a_n, \sigma_u^2, \sigma_w^2\}$. Straightforward calculations show that

$$-\frac{\partial S(z)}{\partial a_k} = \frac{\sigma_u^2 z^k}{a^2(z)a(z^{-1})} + \frac{\sigma_u^2 z^{-k}}{a(z)a^2(z^{-1})}, \tag{8a}$$

$$\frac{\partial S(z)}{\partial \sigma_u^2} = \frac{1}{a(z)a(z^{-1})}, \tag{8b}$$

$$\frac{\partial S(z)}{\partial \sigma_w^2} = 1. \tag{8c}$$

As we will see next, the following quantities are also required.

$$-\frac{\partial S(z)}{\partial a_k} S^{-1}(z) = \frac{\sigma_u^2 z^k}{Ka(z)c(z)c(z^{-1})} + \frac{\sigma_u^2 z^{-k}}{Ka(z^{-1})c(z)c(z^{-1})}, \tag{9a}$$

$$\frac{\partial S(z)}{\partial \sigma_u^2} S^{-1}(z) = \frac{1}{Kc(z)c(z^{-1})}, \tag{9b}$$

$$\frac{\partial S(z)}{\partial \sigma_w^2} S^{-1}(z) = \frac{a(z)a(z^{-1})}{Kc(z)c(z^{-1})}. \tag{9c}$$

2.2. Whittle's formula

Let $S(z)$ be the spectral density function of a discrete time stationary zero-mean scalar process, and assume that this spectral function depends on some parameter vector $\theta = [\theta_1, \ldots, \theta_m]^T$. It was shown by Whittle [10] that the asymptotic form of the Fisher information matrix I_N associated with these parameters is given by

$$I_N = [I_{k,l}], \quad 1 \leq k, l \leq m, \tag{10a}$$

$$I_{k,l} = \frac{N}{4\pi j} \oint \frac{\partial S(z)}{\partial \theta_k} S^{-1}(z) \frac{\partial S(z)}{\partial \theta_l} S^{-1}(z) \frac{dz}{z}, \tag{10b}$$

where $\oint$ represents counter-clockwise integration on the unit circle ($z = e^{j\omega}$), and N is the number of data points used to estimate the parameters θ_i. In the AR-plus-noise case the entries of the Fisher information matrix are given by,

$$\begin{aligned} I_{k,l} &= \frac{N}{2} \frac{1}{2\pi j} \oint \frac{\partial S}{\partial a_k} S^{-1} \frac{\partial S}{\partial a_l} S^{-1} \frac{dz}{z} \\ &= N \left[\frac{1}{2\pi j} \oint \frac{\sigma_u^4 z^{-(k+l)}}{K^2 c^2(z) c^2(z^{-1}) a^2(z^{-1})} \frac{dz}{z} + \frac{1}{2\pi j} \oint \frac{\sigma_u^4 z^{-(k-l)}}{K^2 c^2(z) c^2(z^{-1}) a(z) a(z^{-1})} \frac{dz}{z} \right], \\ & 1 \leq k, l \leq n. \end{aligned} \tag{11}$$

$$\begin{aligned} I_{k,n+1} &= I_{n+1,k} \\ &= \frac{N}{2} \frac{1}{2\pi j} \oint \frac{\partial S}{\partial a_k} S^{-1} \frac{\partial S}{\partial \sigma_u^2} S^{-1} \frac{dz}{z} \\ &= -N \frac{1}{2\pi j} \oint \frac{\sigma_u^2 z^{-k}}{K^2 c^2(z) c^2(z^{-1}) a(z^{-1})} \frac{dz}{z}, \\ & 1 \leq k \leq n. \end{aligned} \tag{12}$$

$$\begin{aligned} I_{k,n+2} &= I_{n+2,k} \\ &= \frac{N}{2} \frac{1}{2\pi j} \oint \frac{\partial S}{\partial a_k} S^{-1} \frac{\partial S}{\partial \sigma_w^2} S^{-1} \frac{dz}{z} \\ &= -N \frac{1}{2\pi j} \oint \frac{\sigma_u^2 a(z) z^{-k}}{K^2 c^2(z) c^2(z^{-1})} \frac{dz}{z}, \\ & 1 \leq k \leq n, \end{aligned} \tag{13}$$

$$\begin{aligned} I_{n+1,n+1} &= \frac{N}{2} \frac{1}{2\pi j} \oint \left[\frac{\partial S}{\partial \sigma_u^2} S^{-1} \right]^2 \frac{dz}{z} \\ &= \frac{N}{2} \frac{1}{2\pi j} \oint \frac{1}{K^2 c^2(z) c^2(z^{-1})} \frac{dz}{z}, \end{aligned} \tag{14}$$

$$\begin{aligned} I_{n+1,n+2} &= I_{n+2,n+1} \\ &= \frac{N}{2} \frac{1}{2\pi j} \oint \frac{\partial S}{\partial \sigma_u^2} S^{-1} \frac{\partial S}{\partial \sigma_w^2} S^{-1} \frac{dz}{z} \\ &= \frac{N}{2} \frac{1}{2\pi j} \oint \frac{a(z)a(z^{-1})}{K^2 c^2(z) c^2(z^{-1})} \frac{dz}{z}, \end{aligned} \tag{15}$$

$$\begin{aligned} I_{n+2,n+2} &= \frac{N}{2} \frac{1}{2\pi j} \oint \left[\frac{\partial S}{\partial \sigma_w^2} S^{-1} \right]^2 \frac{dz}{z} \\ &= \frac{N}{2} \frac{1}{2\pi j} \oint \frac{a^2(z) a^2(z^{-1})}{K^2 c^2(z) c^2(z^{-1})} \frac{dz}{z}. \end{aligned} \tag{16}$$

These expressions can be evaluated by numerical integration. However, if either $a(z)$ or $c(z)$ have roots very close to the unit circle considerable care needs to be exercised to avoid numerical problems. A more attractive way of computing $I_{k,l}$ is described next.

2.3. Evaluation of the complex integrals

To evaluate the integrals in (11)-(16) we must first introduce some notation. Let the polynomial $\gamma(z)$ be defined by

$$\gamma(z) = c^2(z) = 1 + \gamma_1 z + \cdots + \gamma_{2n} z^{2n}. \tag{17}$$

The stability of $\gamma(z)$ follows from the stability of $c(z)$. Next denote

$$\frac{1}{\gamma(z)\gamma(z^{-1})} = \sum_{l=-\infty}^{\infty} v_l z^l; \quad v_l = v_{-l}, \tag{18}$$

$$C_c = \begin{bmatrix} -\gamma_1 & -\gamma_2 & \cdots & -\gamma_{2n} \\ 1 & & 0 & 0 \\ & \ddots & & \vdots \\ 0 & & 1 & 0 \end{bmatrix};$$

$$\zeta = \begin{bmatrix} v_0 \\ v_{-1} \\ \vdots \\ v_{-2n+1} \end{bmatrix}; \quad e_{2n} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

Then it can be shown [13] that

$$\begin{aligned} v_l &= e_{2n}^{\mathrm{T}}(C_c)^{l+2n-1}\zeta \\ &= \zeta^{\mathrm{T}}(C_c^{\mathrm{T}})^{l+2n-1}e_{2n}; \quad l \geq -2n+1. \end{aligned} \tag{19}$$

Similary, we denote

$$\frac{1}{a(z)a(z^{-1})} = \sum_{l=-\infty}^{\infty} r_l z^l; \quad r_l = r_{-l} \tag{20}$$

$$A_c = \begin{bmatrix} -a_1 & -a_2 & \cdots & \cdots & -a_n \\ 1 & & 0 & & 0 \\ & \ddots & & & \vdots \\ & 0 & & 1 & 0 \end{bmatrix};$$

$$\rho = \begin{bmatrix} r_0 \\ r_{-1} \\ \vdots \\ r_{-n+1} \end{bmatrix}; \quad e_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}. \tag{21}$$

Then

$$\begin{aligned} r_l &= e_n^{\mathrm{T}}(A_c)^{l+n-1}\rho \\ &= \rho^{\mathrm{T}}(A_c^{\mathrm{T}})^{l+n-1}e_n, \quad l \geq -n+1. \end{aligned} \tag{22}$$

Next define,

$$\frac{1}{a(z^{-1})} = \sum_{m=0}^{\infty} k_m z^{-m}. \tag{23}$$

Then

$$\begin{aligned} k_m &= e_1^{\mathrm{T}}(A_c)^m e_1 \\ &= e_1^{\mathrm{T}}(A_c^{\mathrm{T}})^m e_1, \quad m \geq 0, \end{aligned} \tag{24}$$

where $e_1 = [1, 0, \ldots, 0]^{\mathrm{T}}$ = an n-dimensional unit vector. Finally, let

$$\alpha(z) = a^2(z) = 1 + \alpha_1 z + \cdots \alpha_{2n} z^{2n}. \tag{25}$$

$$B_c = \begin{bmatrix} -\alpha_1 & -\alpha_2 & \cdots & -\alpha_{2n} \\ 1 & & 0 & 0 \\ & \ddots & & \vdots \\ & 0 & 1 & 0 \end{bmatrix},$$

$$\frac{1}{\alpha(z^{-1})} = \sum_{m=0}^{\infty} g_m z^{-m}. \tag{26}$$

Then

$$g_m = \bar{e}_1^{\mathrm{T}}(B_c^{\mathrm{T}})^m \bar{e}_1; \quad m \geq 0, \tag{27}$$

where $\bar{e}_1 = [0, 1, \ldots, 0]^{\mathrm{T}}$ = a $2n$-dimensional unit vector.

Using the quantities $\{v_l, r_l, h_l, g_l\}$ defined above, we can now evaluate the various complex integrals introduced earlier. The first integral in (11) is just the coefficient of z^{k+l} in the power series expansion of

$$\frac{1}{\gamma(z)\gamma(z^{-1})\alpha(z^{-1})}.$$

Note that,

$$\begin{aligned} \frac{1}{\gamma(z)\gamma(z^{-1})} \cdot \frac{1}{\alpha(z^{-1})} &= \sum_{l=-\infty}^{\infty} v_l z^l \cdot \sum_{m=0}^{\infty} g_m z^{-m} \\ &= \sum_{l=-\infty}^{\infty} \sum_{m=0}^{\infty} v_l g_m z^{l-m}. \end{aligned} \tag{28}$$

Thus,

$$\begin{aligned} &\frac{1}{2\pi j}\oint \frac{z^{-(k+l)}}{\gamma(z)\gamma(z^{-1})\alpha(z^{-1})}\frac{dz}{z} \\ &= \sum_{m=0}^{\infty} v_{k+l+m} g_m \\ &= \sum_{m=0}^{\infty} \zeta^{\mathrm{T}}(C_c^{\mathrm{T}})^{k+l+2n-1+m} e_{2n}\bar{e}_1^{\mathrm{T}}(B_c^{\mathrm{T}})^m \bar{e}_1 \\ &= \zeta^{\mathrm{T}}(C_c^{\mathrm{T}})^{k+l+2n-1} \\ &\quad \times \left[\sum_{m=0}^{\infty} (C_c^{\mathrm{T}})^m e_{2n}\bar{e}_1^{\mathrm{T}}(B_c^{\mathrm{T}})^m\right]\bar{e}_1 \\ &= \zeta^{\mathrm{T}}(C_c^{\mathrm{T}})^{k+l+2n-1} GS\bar{e}_1 \\ &= [v_{k+l+2n-1}, \ldots, v_{k+l}]GS\bar{e}_1, \end{aligned} \tag{29}$$

where the matrices S and G are defined in eqs. (A1) and (A3) of the Appendix.

The second integral in (11) is the coefficient of z^{k-l} in the power series expansion of

$$\frac{1}{\gamma(z)\gamma(z^{-1})a(z)a(z^{-1})}.$$

Note that,

$$\frac{1}{\gamma(z)\gamma(z^{-1})a(z)a(z^{-1})} = \sum_{l=-\infty}^{\infty} v_l z^l \sum_{m=-\infty}^{\infty} r_m z^{-m} = \sum_{l=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} v_l r_m z^{l-m} \tag{30}$$

Hence

$$\begin{aligned}\frac{1}{2\pi j}\oint \frac{z^{-(k-l)}}{\gamma(z)\gamma(z^{-1})a(z)a(z^{-1})}\frac{dz}{z}
&= \sum_{m=-\infty}^{\infty} v_{k-l+m} r_m \\
&= \sum_{m=0}^{\infty} r_m v_{k-l-m} + \sum_{m=-\infty}^{-1} r_m v_{k-l+m} \\
&= \sum_{m=0}^{\infty} r_m v_{k-l+m} + \sum_{m=1}^{\infty} r_{-m} v_{k-l-m} \\
&= \sum_{m=0}^{\infty} r_m v_{k-l+m} + \sum_{m=1}^{\infty} r_m v_{m+l-k} \\
&= \sum_{m=0}^{\infty} r_m v_{m+k-l} + \sum_{m=0}^{\infty} r_m v_{m+l-k} - r_0 v_{l-k} \\
&= \sum_{m=0}^{\infty} \zeta^{\mathrm{T}}(C_c^{\mathrm{T}})^{2n-1+k-l+m} e_{2n} e_n^{\mathrm{T}}(A_c)^{m+n-1}\rho \\
&\quad + \sum_{m=0}^{\infty} \zeta^{\mathrm{T}}(C_c^{\mathrm{T}})^{2n-1+l-k+m} \times e_{2n} e_n^{\mathrm{T}}(A_c)^{m+n-1}\rho - r_0 v_{l-k} \\
&= \zeta^{\mathrm{T}}(C_c^{\mathrm{T}})^{2n-1-k-l} \times \left[\sum_{m=0}^{\infty}(C_c^{\mathrm{T}})^m e_{2n} e_n^{\mathrm{T}}(A_c)^m\right](A_c)^{n-1}\rho \\
&\quad + \zeta^{\mathrm{T}}(C_c^{\mathrm{T}})^{2n-1+l-k} \times \left[\sum_{m=0}^{\infty}(C_c^{\mathrm{T}})^m e_{2n} e_n^{\mathrm{T}}(A_c)^m\right] \times (A_c)^{n-1}\rho - r_0 v_{l-k} \\
&= [v_{2n-1+k-l}, \ldots, v_{k-l}]G \times UH[r_{n-1}, \ldots, r_0]^{\mathrm{T}} \\
&\quad + [v_{2n-1+l-k}, \ldots, v_{l-k}]G \times UH[r_{n-1}, \ldots, r_0]^{\mathrm{T}} - r_0 v_{l-k},
\end{aligned} \tag{31}$$

where U and H are defined in eqs. (A8) and (A9) of the Appendix.

The integral in (12) can be evaluated similarly to the integral in (29), using $a(z^{-1})$ instead of $\alpha(z^{-1})$. The result of this evaluation is

$$\frac{1}{2\pi j}\oint \frac{z^{-k}}{\gamma(z)\gamma(z^{-1})a(z^{-1})}\frac{dz}{z} = \zeta^{\mathrm{T}}(C_c^{\mathrm{T}})^{k+2n-1} GUe_1 = [v_{k-2n-1}, \ldots, v_k]GUe_1. \tag{32}$$

It is straightforward to check that the integral in (13) is given by

$$\frac{1}{2\pi j}\oint \frac{a(z)z^{-k}}{\gamma(z)\gamma(z^{-1})}\frac{dz}{z} = \sum_{l=0}^{n} a_l v_{l-k}; \quad a_0 = 1. \tag{33}$$

The integral in (14) is given by,

$$\frac{1}{2\pi j}\oint \frac{1}{\gamma(z)\gamma(z^{-1})}\frac{dz}{z} = v_0, \tag{34}$$

and the integral in (15) is given by

$$\frac{1}{2\pi j}\oint \frac{a(z)a(z^{-1})}{\gamma(z)\gamma(z^{-1})}\frac{dz}{z} = \sum_{l=0}^{n}\sum_{m=0}^{n} a_l a_m v_{l-m}.$$

Finally, the integral in (10) is given by

$$\frac{1}{2\pi j}\oint \frac{a^2(z)a^2(z^{-1})}{\gamma(z)\gamma(z^{-1})}\frac{dz}{z} = \sum_{l=0}^{2n}\sum_{m=0}^{2n} \alpha_l \alpha_m v_{l-m}. \tag{36}$$

Eqs. (29), (31)–(38) provide explicit expressions for computing the entries of the Fisher information matrix. The computation of the quantities appearing in these equations (the scalars v_l, r_l and the matrices G, S, U, H) is discussed in the Appendix.

3. Bounds on the estimates of spectral parameters

In the previous section we developed the asymptotic bounds on variance of unbiased estimates of the AR parameters $\theta = [a_1, \ldots, a_m, \sigma_u^2, \sigma_w^2]^T$. In many applications one is interested not in the AR parameters, but in some function of these parameters, such as the spectrum of the signal-plus-noise (cf. (3)) or the signal only,

$$S_v(z) = \frac{\sigma_u^2}{a(z)a(z^{-1})}. \tag{37}$$

In the case of signals with narrowband spectra we may be interested in spectral parameters such as bandwidth, center-frequency and power of each narrowband component. In this section we derive the formulas for computing the bound on the estimation error of various functions of the AR-plus-noise parameters.

3.1. A general formula

Given a scalar function $f(\theta)$ of a parameter vector $\theta = [\theta_1, \ldots, \theta_m]^T$, the variance of any unbiased estimator of $f(\theta)$ from N data points is bounded from below by the following generalized Cramer-Rao bound [11]:

$$\mathrm{Var}\{\hat{f}(\theta)\} \geq D^T I_N^{-1} D, \tag{38a}$$

where

$$D^T = \left[\frac{\partial f(\theta)}{\partial \theta_1}, \ldots, \frac{\partial f(\theta)}{\partial \theta_m}\right]$$

$$= \text{vector of partial derivatives}, \tag{38b}$$

I_N = the Fisher information

matrix associated with estimating

θ (cf. (10)). (38c)

The computation of I_N was discussed in detail in Section 2. It remains to evaluate the derivative vector D for the functions of interest.

3.2. The signal-plus-noise spectrum

To evaluate the bounds on the signal-plus-noise spectrum $S(e^{j\omega})$ we must evaluate the entries of

$$D_1^T \triangleq \left[\frac{\partial S(e^{j\omega})}{\partial a_1}, \ldots, \frac{\partial S(e^{j\omega})}{\partial a_n}, \frac{\partial S(e^{j\omega})}{\partial \sigma_u^2}, \frac{\partial S(e^{j\omega})}{\partial \sigma_w^2}\right]. \tag{39}$$

Straightforward differentiation of equation (3) gives the following:

$$\frac{\partial S(e^{j\omega})}{\partial a_k} = -\frac{2\sigma_u^2}{a(e^{j\omega})a(e^{-j\omega})}\,\mathrm{Re}\left\{\frac{e^{jk\omega}}{a(e^{j\omega})}\right\}, \tag{40a}$$

$$\frac{\partial S(e^{j\omega})}{\partial \sigma_u^2} = \frac{1}{a(e^{j\omega})a(e^{-j\omega})}, \tag{40b}$$

$$\frac{\partial S(e^{j\omega})}{\partial \sigma_w^2} = 1. \tag{40c}$$

To evaluate the bound on the signal spectrum $S_v(e^{j\omega})$ we must use

$$D_2^T \triangleq \left[\frac{\partial S(e^{j\omega})}{\partial a_1}, \ldots, \frac{\partial S(e^{j\omega})}{\partial a_n}, \frac{\partial S(e^{j\omega})}{\partial \sigma_u^2}, 0\right]. \tag{41}$$

3.3. Spectral parameters of a second order AR-plus-noise process

Consider a second order AR process with a polynomial having a complex pair of roots at $\rho e^{\pm j\omega_0}$,

$$a(z) = 1 + a_1 z + a_2 z^2$$
$$= 1 - 2\rho\cos\omega_0 z + \rho^2 z^2. \tag{42}$$

The central frequency f_0 of the spectrum of this process is defined by the angle (or phase) of the roots:

$$f_0 = \omega_0 / 2\pi. \tag{43}$$

To compute the bound on the estimation error of f_0 we must evaluate

$$D_3^T \triangleq \left[\frac{\partial f_0}{\partial a_1}, \frac{\partial f_0}{\partial a_2}, \frac{\partial f_0}{\partial \sigma_u^2}, \frac{\partial f_0}{\partial \sigma_w^2}\right]. \tag{44}$$

The partial derivatives are given by

$$\frac{\partial f_0}{\partial a_1}=\frac{1}{2\pi\sqrt{4a_2-a_1^2}}, \tag{45}$$

$$\frac{\partial f_0}{\partial a_2}=\frac{-a_1}{4\pi a_2\sqrt{4a_2-a_1^2}}. \tag{46}$$

$$\frac{\partial f_0}{\partial \sigma_u^2}=\frac{\partial f_0}{\partial \sigma_w^2}=0. \tag{47}$$

Another frequency of interest is the frequency $\bar{f}$ for which $S(e^{j\omega})$ is maximized. Note that in the second order case

$$S_x(e^{j\omega})=\frac{\sigma_u^2}{(1+a_1^2+a_2^2)+2a_1(1+a_2)\cos\omega+2a_2\cos 2\omega}. \tag{48}$$

The frequency at which $S_x(e^{j\omega})$ is maximum is the frequency at which $1/S_x(e^{j\omega})$ is minimum. We find this frequency by setting the derivative to zero:

$$\begin{aligned}&\frac{\partial}{\partial\omega}(1/S_x(e^{j\omega}))\\ &\quad=-2\sin\omega[a_1(1+a_2)+4a_2\cos\omega]\\ &\quad=0.\end{aligned} \tag{49}$$

The points $\omega=0$ and $\omega=\pi$ correspond to minima of $S_x(\omega)$. The maximum is attained at

$$\bar{f}=\frac{1}{2\pi}\cos^{-1}\left(-\frac{a_1(1+a_2)}{4a_2}\right). \tag{50}$$

In this case

$$D_4^T=\left[\frac{\partial\bar{f}}{\partial a_1},\frac{\partial\bar{f}}{\partial a_2},0,0\right], \tag{51}$$

where

$$\frac{\partial\bar{f}}{\partial a}=\frac{1}{2\pi\sin\bar{\omega}}\,\frac{1+a_2}{4a_2}, \tag{52}$$

$$\frac{\partial\bar{f}}{\partial a_2}=-\frac{1}{2\pi\sin\bar{\omega}}\,\frac{a_1}{4a_2^2}. \tag{53}$$

Another spectral parameter of practical interest is the bandwidth of the spectrum. We will use the so-called noise bandwidth defined by

$$B=\frac{\dfrac{1}{2\pi}\displaystyle\int_0^{\pi}S_x(e^{j\omega})\,d\omega}{S_x(e^{j\bar{\omega}})}. \tag{54}$$

The numerator is the energy contained in the positive frequencies, while the denominator is the peak energy density. The resulting B is normalized with respect to 0.5 Hz, (i.e., for white noise, $B=0.5$ Hz). Note that

$$\begin{aligned}&\frac{1}{2\pi}\int_0^{\pi}S_x(e^{j\omega})\,d\omega\\ &\quad=\frac{(1+a_2)}{2(1-a_2)(1+a_2+a_1)(1+a_2-a_1)},\end{aligned} \tag{55}$$

$$S_x(e^{j\bar{\omega}})=\frac{4a_2}{(1-a_2)^2(4a_2-a_1^2)}. \tag{56}$$

Thus,

$$B=\frac{(1+a_2)(1-a_2)(4a_2-a_1^2)}{8a_2(1+a_2+a_1)(1+a_2-a_1)}. \tag{57}$$

To evaluate the bound on the error variance of $\hat{B}$ we need to compute

$$D_5^T=\left[\frac{\partial B}{\partial a_1},\frac{\partial B}{\partial a_2},0,0\right]. \tag{58}$$

Note that

$$\frac{\partial\log B}{\partial a_1}=\frac{-2a_1}{4a_2-a_1^2}-\frac{1}{1+a_2+a_1}+\frac{1}{1+a_2-a_1}, \tag{59a}$$

$$\begin{aligned}\frac{\partial\log B}{\partial a_2}&=\frac{1}{1+a_2}-\frac{1}{1-a_2}+\frac{4}{4a_2-a_1^2}\\ &\quad-\frac{1}{a_2}-\frac{1}{1+a_2+a_1}-\frac{1}{1+a_2-a_1}.\end{aligned} \tag{59b}$$

Thus,

$$\frac{\partial B}{\partial a_1}=\frac{\partial\log B}{\partial a_1}\cdot B, \tag{60a}$$

$$\frac{\partial B}{\partial a_2}=\frac{\partial\log B}{\partial a_2}\cdot B. \tag{60b}$$

Finally, we consider the signal power, defined as

$$P=\frac{1}{2\pi}\oint\frac{\sigma_u^2}{a(z)a(z^{-1})}\frac{dz}{z}. \tag{61}$$

For second-order AR process, the complex integral yields

$$P=\frac{\sigma_u^2(1+a_2)}{(1-a_2)(1+a_2+a_1)(1+a_2-a_1)}. \tag{62}$$

To evaluate the bound on $\hat{P}$ we must compute

$$D_6^T=\left[\frac{\partial P}{\partial a_1},\frac{\partial P}{\partial a_2},\frac{\partial P}{\partial \sigma_u^2},0\right]. \tag{63}$$

Note that

$$\frac{\partial \log P}{\partial a_2}=\frac{1}{1+a_2-a_1}-\frac{1}{1+a_2+a_1},$$

$$\frac{\partial \log P}{\partial a_1}=\frac{1}{1+a_2}+\frac{1}{1-a_2}-\frac{1}{1+a_2+a_1}-\frac{1}{1+a_2-a_1}, \tag{64}$$

thus.

$$\frac{\partial P}{\partial a_1}=\frac{\partial \log P}{\partial a_1}P, \tag{65a}$$

$$\frac{\partial P}{\partial a_2}=\frac{\partial \log P}{\partial a_2}P, \tag{65b}$$

and finally,

$$\frac{\partial P}{\partial \sigma_u^2}=\frac{1}{\sigma_u^2}P. \tag{66}$$

Inserting D_i in (38a) gives an explicit formula for computing the CRLB for various spectral parameters of practical interest.

4. Some examples

In this section we present a few examples illustrating the usefulness of the bounds derived in the previous sections. Two AR models are considered:

S_1: $a(z)=1-1.4z+0.95z^2$ (narrowband)

S_2: $a(z)=1-0.45z+0.55z^2$ (broadband)

4.1. Spectral bounds

Using the derivative vector D_1 (39) we computed the CRLB for the signal-plus-noise spectrum for S_1 and S_2, at different frequencies. The true spectrum and the ± 1 standard deviation curves are depicted in Fig. 1 and 2. These plots provide some insight into the achievable spectral estimation accuracy for the given signal and noise parameters.

4.2. Bounds on center frequency, bandwidth and power

Using the derivative vectors in equations (51), (58) and (63) we computed the CRLB for $\bar{f}$, B and P for S_1 and S_2, at different signal-to-noise ratios. The results are summarized in Tables 1 and 2.

Examination of these tables reveals various interesting facts. In the narrowband case the center frequency can be estimated much more accurately than bandwidth and power. Note for example that at SNR = 3 dB the relative accuracy (i.e., standard deviation divided by the mean) of $\bar{f}$ is 0.8%, of B 25% and of P 20%. The situation is similar in the broadband case. However, the center frequency is estimated less accurately than in the narrowband case. For example, at SNR = 3 dB the relative accuracies of $\bar{f}$, B and P are 2.4%, 15% and 16%. This type of behavior has been observed in simulation studies of various parametric spectral estimation techniques.

Table 1

Bounds on the standard deviation of the estimates of $\bar{f}$, B and P. $a(z)=1-1.4z^{-1}+0.95z^{-2}$; $\sigma_u^2=0.04709$; $N=1024$

SNR (dB)	$\bar{f}=0.1224$	$B=0.01281$	$P=1.0$
+3	$0.9440\cdot10^{-3}$	$0.3153\cdot10^{-2}$	0.2005
0	$0.1016\cdot10^{-2}$	$0.3473\cdot10^{-2}$	0.2068
−3	$0.1128\cdot10^{-2}$	$0.3970\cdot10^{-2}$	0.2200
−6	$0.1309\cdot10^{-2}$	$0.4772\cdot10^{-2}$	0.2482
−9	$0.1612\cdot10^{-2}$	$0.6125\cdot10^{-2}$	0.3072
−12	$0.2147\cdot10^{-2}$	$0.8528\cdot10^{-2}$	0.4278
−15	$0.3137\cdot10^{-2}$	$0.1299\cdot10^{-1}$	0.6691

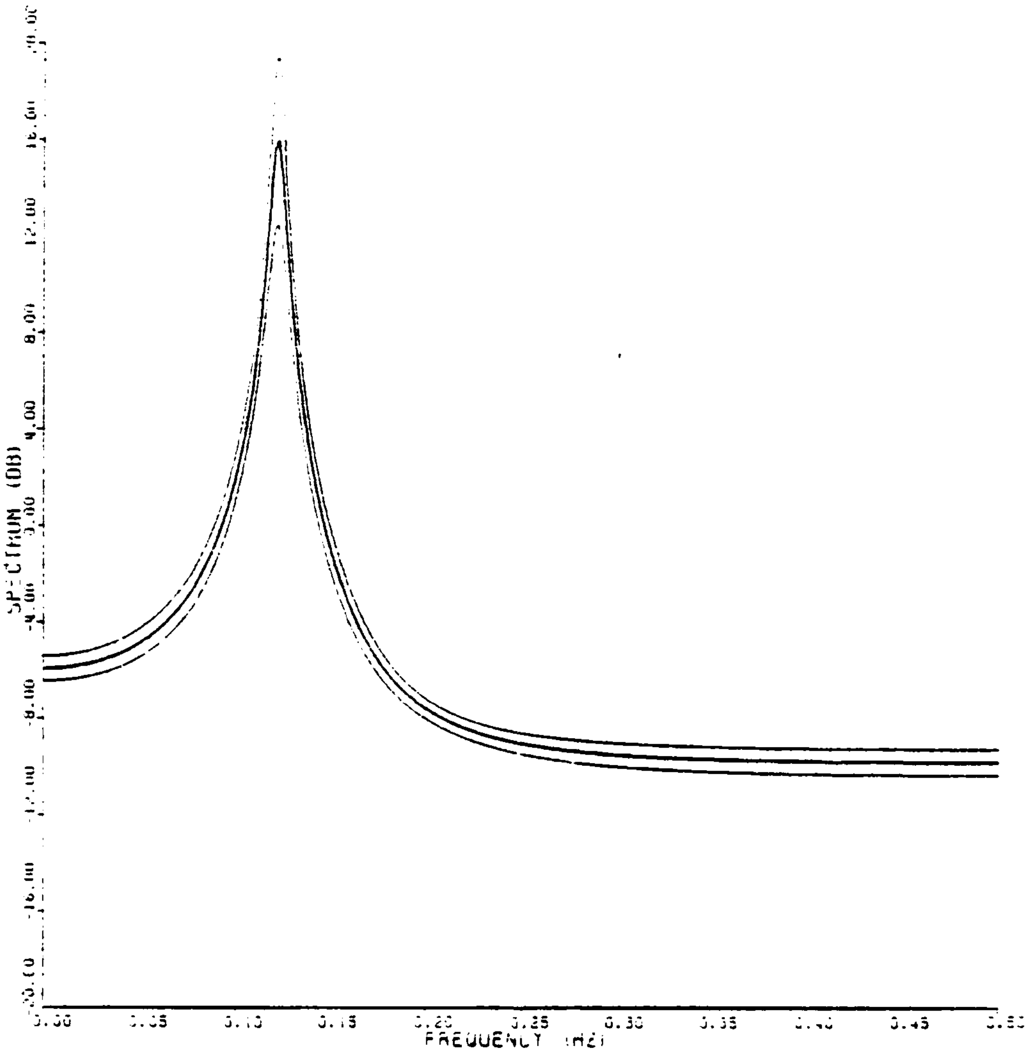

Fig. 1. Spectral bound for S_1, $\sigma_u^2 = 0.04709$, $\sigma_n^2 = 0.1$, SNR = 10 dB, $N = 256$.

Table 2

Bounds on the standard deviation of the estimates of f, B and P. $a(z) = 1 - 0.45z^{-1} + 0.55z^{-2}$; $\sigma_u^2 = 0.6384$; $N = 1024$

SNR (dB)	$\bar{f} = 0.1986$	$B = 0.1439$	$P = 1.0$
−3	$0.4792 \cdot 10^{-2}$	$0.2257 \cdot 10^{-1}$	0.1154
0	$0.6091 \cdot 10^{-2}$	$0.2940 \cdot 10^{-1}$	0.1607
−3	$0.8486 \cdot 10^{-2}$	$0.4227 \cdot 10^{-1}$	0.2493
−6	$0.1303 \cdot 10^{-1}$	$0.6698 \cdot 10^{-1}$	0.4224
−9	$0.2188 \cdot 10^{-1}$	0.1153	0.7637

5. Conclusions

We presented formulas for computing the CRLB for different spectral parameters of an AR-plus-noise process. The proposed formulas make it possible to compute the CRLB without requiring numerical integration. These bounds provide a useful reference point for the performance evaluation of autoregressive estimation techniques.

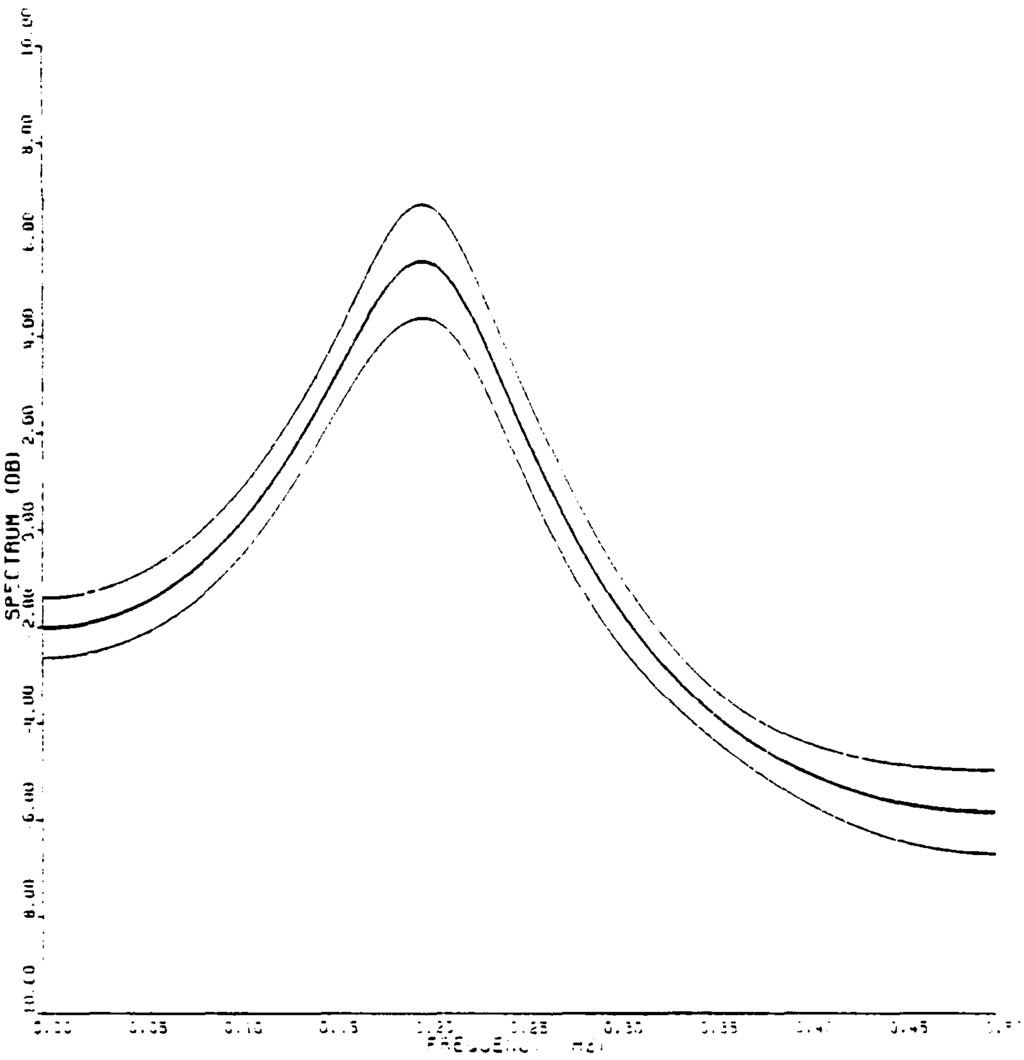

Fig. 2. Spectral bound for S_2. $\sigma_u^2 = 0.6384$, $\sigma_v^2 = 0.1$, SNR = 10 dB, $N = 256$.

Appendix: Some Lyapunov equations

Let

$$S = \sum_{m=0}^{\infty} (C_c)^m \bar{e}_1 \bar{e}_1^{\mathrm{T}} (B_c^{\mathrm{T}})^m. \tag{A1}$$

S clearly satisfies the Lyapunov equation

$$S - C_c S B_c^{\mathrm{T}} = \bar{e}_1 \bar{e}_1^{\mathrm{T}}, \tag{A2}$$

where $\bar{e}_1 = [1, 0, \ldots, 0]^{\mathrm{T}}$ is a $2n$-dimensional unit vector. Moreover, stability of the polynomials $\gamma(z)$ and $\alpha(z)$ guarantees that (A1) is the unique solution of (A2) [15].

Let G be the matrix

$$G = \begin{bmatrix} & & & & 1 \\ & 0 & & \cdot & \gamma_1 \\ & & \cdot & & \vdots \\ 1 & & \gamma_1 & \cdots & \gamma_{2n-1} \end{bmatrix} \tag{A3}$$

It can be checked by direct computation that

$$GC_c = C_c^{\mathrm{T}}G. \tag{A4}$$

Hence:

$$\begin{aligned} GS - C_c^{\mathrm{T}}GSB_c^{\mathrm{T}} &= GS - GC_cSB_c^{\mathrm{T}} \\ &= G(S - C_cSB_c^{\mathrm{T}}) \\ &= G\bar{e}_1\bar{e}_1^{\mathrm{T}} = \bar{e}_{2n}\bar{e}_1^{\mathrm{T}}. \end{aligned} \tag{A5}$$

It follows that,

$$GS = \sum_{m=0}^{\infty}(C_c^{\mathrm{T}})^m e_{2n}\bar{e}_1^{\mathrm{T}}(B_c^{\mathrm{T}})^m. \tag{A6}$$

Let

$$U = \sum_{m=0}^{\infty}(C_c)^m\bar{e}_1 e_1^{\mathrm{T}}(A_c^{\mathrm{T}})^m. \tag{A7}$$

where $e_1 = [1, 0, \ldots, 0]^{\mathrm{T}}$ = an n-dimensional unit vector. U satisfies

$$U - C_cUA_c^{\mathrm{T}} = \bar{e}_1 e_1^{\mathrm{T}} \tag{A8}$$

Let H be the matrix

$$H = \begin{bmatrix} & & & 1 \\ & 0 & \cdot & a_1 \\ & \cdot & \cdot & \vdots \\ 1 & a_1 & \cdots & a_{n-1} \end{bmatrix}. \tag{A9}$$

Then

$$HA_c = A_c^{\mathrm{T}}H. \tag{A10}$$

Hence:

$$\begin{aligned} GUH - C_c^{\mathrm{T}}GUHA_c &= GUH - GC_cUA_c^{\mathrm{T}} \\ &= G(U - C_cUA_c^{\mathrm{T}})H \\ &= G\bar{e}_1 e_1^{\mathrm{T}}H = e_{2n}e_n^{\mathrm{T}}, \end{aligned} \tag{A11}$$

and finally,

$$GUH = \sum_{m=0}^{\infty}(C_c^{\mathrm{T}})^m e_{2n}e_n^{\mathrm{T}}(A_c)^m.$$

The numerical solution of the Lyapunov equations (A2) and (A8) can be performed efficiently using the algorithms suggested in [16] and [17]. It is worthwhile to note that S and U are (nonsymmetric) Toeplitz matrices. Thus, their entries are fully determined by their first row and column.

To compute $\{v_l\}$ we solve the Lyapunov equation

$$X - C_cXC_c^{\mathrm{T}} = \bar{e}_1\bar{e}_1^{\mathrm{T}}, \tag{A12}$$

where X is a Toeplitz matrix. The first $2n$ terms of $\{v_l\}$ are the entries of the first column of X. Higher order terms of $\{v_l\}$ are obtained from the recursion

$$v_l = -\sum_{i=1}^{2n}\gamma_i v_{l-i}, \quad l \geq 2n. \tag{A13}$$

Similarly, the first n terms $\{r_l\}$ are the entries of the first column (or row) of the Toeplitz matrix Y, where

$$Y - A_cYA_c^{\mathrm{T}} = e_1e_1^{\mathrm{T}}. \tag{A14}$$

Higher order terms of $\{r_l\}$ are obtained from the recursion

$$r_l = -\sum_{i=1}^{n}a_i r_{l-i}, \quad l \geq n. \tag{A15}$$

These formulas are explained in more detail in [13].

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APPENDIX B

OPTIMAL INSTRUMENTAL VARIABLE ESTIMATES OF THE AR PARAMETERS OF AN ARMA PROCESS

Optimal Instrumental Variable Estimates of the AR Parameters of an ARMA Process

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Abstract—**The modified Yule-Walker (MYW) equations for estimating the AR parameters of an ARMA process are presented as a special case of an instrumental variable (IV) method. The consistency and accuracy of the AR parameter estimates are studied. It is shown that estimation accuracy increases monotonically with the number of MYW equations for an optimal choice of the weighting matrix used in the least-squares solution of these equations. The asymptotic error covariance of the optimal IV method equals that of the prediction error method. The results of this paper verify experimental results reported in the literature regarding the performance of the MYW method, and provide the necessary accuracy analysis. Furthermore, they suggest several simple, asymptotically efficient, multistep algorithms for estimating the AR parameters, which are presented in a companion paper.**

I. Introduction

The need for estimating the parameters of an autoregressive moving-average (ARMA) process arises in many applications in the areas of signal processing, spectral analysis, and system identification. A computationally attractive estimation procedure, which has received considerable attention in the literature, is based on a two-step approach: first the autoregressive (AR) parameters are estimated using the modified Yule-Walker (MYW) equations; then the moving average (MA) parameters are estimated by one of several available techniques.

In this paper we consider only the first step of estimating the autoregressive parameters. In many engineering applications the second estimation step is not needed. The prime example is the estimation of autoregressive signals corrupted by white measurement noise. In this case all the information about the spectral shape of the signal lies in the AR parameters of the signal-plus-noise ARMA process (see, e.g., [29]).

The relative simplicity of the MYW estimator motivated a number of authors to investigate this technique and to develop various extensions and variations [1]-[10]. Most of this work has been done in the context of high resolution spectral analysis. One of the important observations made in these studies is that significant improvements in estimation accuracy can be obtained by increasing the number of MYW equations [2], [9]. The resulting set of overdetermined equations is then solved by some least-squares technique. The possibility of using a weighted least-squares procedure was also discussed (see, e.g., [1], [2]).

Performance evaluation of the MYW method has in the past been done by simulation. A formal accuracy theory appears to be lacking. It is our objective in this paper to fill this gap and provide an asymptotic accuracy analysis. This analysis clarifies the precise role of increasing the number of equations and of including a weighting matrix. It provides a valuable verification for experimental observations as well as guidelines for further improvements of MYW based ARMA estimation techniques.

The MYW method is related to the instrumental variable (IV) method of parameter estimation [8], [11], [12]. In Section II we define an IV estimator which is slightly more general than the MYW estimators presented in the literature. In Section III we establish the consistency of the IV estimates and develop an explicit formula for the covariance matrix of the estimation errors. This formula can be used to evaluate the asymptotic performance of various MYW algorithms proposed in the literature [23], [30]. In Section IV we study the optimization of estimation accuracy with respect to the weighting matrix and the number of equations. We show the existence of an optimal choice of the weighting matrix, which minimizes the covariance matrix of the estimation errors. Furthermore, we show that the optimal error covariance matrix decreases monotonically when the number of equations is increased, and converges as the number of equations tends to infinity. The form of this limiting matrix is also presented, and in Section V it is shown that it equals the asymptotic error covariance of the prediction error method. The effect of a certain filter used in the generation of the instrumental variables on the convergence rate of the error covariance matrix of the optimally weighted IV estimate is studied in Section VI. It is shown that there exists an optimal choice of this filter which gives the fastest convergence rate.

The optimal IV methods presented in this paper can be used to derive several new AR parameter estimation algorithms with improved accuracy and modest computational cost. In a companion paper [24] we present several such algorithms, analyze their asymptotic properties, and evaluate their performance by simulation.

Finally, we note that results related to those presented here appeared recently in [31]-[33]. The problem considered in these references is the estimation of the parameters of dynamic econometric models by IV methods with instruments that are not exogenous. The approach used in [31]-[33] is based on a different formalism from the one used here.

II. The Estimation Method

Consider the following ARMA process of order (na, nc):

$$A(q^{-1})y(t)=C(q^{-1})e(t) \tag{1}$$

where $e(t)$ is a white noise process with zero mean and variance λ^2, and

$$A(q^{-1})=1+a_1q^{-1}+\cdots+a_{na}q^{-na},$$

$$C(q^{-1})=1+c_1q^{-1}+\cdots+c_{nc}q^{-nc},$$

$$q^{-1}=\text{unit delay operator } (q^{-1}y(t)=y(t-1)).$$

The following assumptions are made:

A1: $A(z)=0 \Rightarrow |z|>1$; $C(z)=0 \Rightarrow |z|>1$. In other words, the ARMA representation (1) is stable and invertible. This is not a restrictive assumption (cf. the spectral factorization theorem, e.g., [28]).

Manuscript received January 26, 1984; revised March 28, 1985. Paper recommended by Past Associate Editor, A. S. Willsky. The work of B. Friedlander was supported by the Army Research Office under Contract DAAG29-83-C-0027.

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where

$$S = E\{[C(q^{-1})z(t)][C(q^{-1})z(t)]^T\} \tag{15}$$

with $\hat{\theta}$, R defined by (6) and (11). ∎

Theorem 1 can be used to evaluate various choices of Q, $G(\cdot)$, and m by comparing the accuracies of the resulting estimates. In [30] we evaluated P for some low-order ARMA systems and various choices of Q and m (with $G(q^{-1}) \equiv 1$). It was observed that accuracy does not increase monotonically with m, in contrast with statements sometimes made in the literature on the overdetermined MYW equations [2], [3]. Furthermore, it appears difficult, if not impossible, to predict which ad-hoc weighting matrix Q will lead to best accuracy.

We have also compared the accuracy of the IV estimate to that given by the prediction error method (PEM) [18], [19], [26], for some simple low-order systems; see, e.g., Example 1 in Section V and the examples in [23]. Recall that in the Gaussian case, the PEM error covariance matrix equals the Cramér-Rao lower bound. The differences in accuracy between the IV method and the PEM were sometimes considerable, indicating that the IV estimator with ad-hoc choices of Q, m, and $G(q^{-1})$ is inefficient (in the statistical sense).

The questions raised above motivate the more detailed examination of the accuracy aspects of the IV estimates. In particular, it is of interest to choose Q, m, and $G(q^{-1})$ so as to increase the accuracy of the IV estimate (6). This is discussed in Sections IV–VI.

IV. Optimization of Estimation Accuracy

The problem of determining optimal IV estimates in the fairly general class of estimates defined by (6) can be stated as follows. Find Q_{opt}, m_{opt}, and $G_{opt}(q^{-1})$ such that the corresponding covariance matrix P_{opt} has the property $P \geq P_{opt}$, where P corresponds to any other admissible choice of Q, m, and G. This type of problem was studied in [12], [13] for systems with exogenous inputs, such as ARMAX systems. The results of [12], [13] cannot be applied directly to the ARMA problem, as is explained in [30]. Therefore, we must approach the accuracy optimization in another way. As we will see, the optimization with respect to Q, m, and $G(q^{-1})$ can be treated in three distinct steps. We start with the optimization of P given by (14) with respect to the weighting matrix Q, for which the following result holds.

Theorem 2: Consider the matrix P defined in (14). We have

$$P \geq (R^TS^{-1}R)^{-1} \triangleq \bar{P}_m. \tag{16}$$

Furthermore, the equality $P = \bar{P}_m$ holds if and only if

$$SQR = R(R^TS^{-1}R)^{-1}(R^TQR). \tag{17}$$

Proof: It is straightforward to show that

$$P - \bar{P}_m = [(R^TQR)^{-1}R^TQ - (R^TS^{-1}R)^{-1}R^TS^{-1}] \cdot S[(R^TQR)^{-1}R^TQ - (R^TS^{-1}R)^{-1}R^TS^{-1}]^T. \tag{18}$$

Since $S > 0$, (16) and (17) follow. ∎

Note that (16) is closely related to the Gauss–Markov theorem in regression theory [22]. *An obvious way to satisfy (17) is to set $Q = S^{-1}$, in which case $P = \bar{P}_m$.*

Next we consider the optimization of $\bar{P}_m$ with respect to m. In Section VI (Lemma 2) we will formally prove that *for the optimal choice of Q, estimation accuracy increases monotonically with m, i.e., $\bar{P}_m \geq \bar{P}_{m+1}$ for all $m \geq na$.* As was mentioned earlier, this is not true for arbitrary choices of Q [23], [30].

Note that the results above are valid for general IV estimation problems. The detailed structure of the matrices R and S is not used anywhere in the proofs. Note also that for AR systems it can be shown that $\bar{P}_{m+1} = \bar{P}_m (m \geq na)$ [21]. However, for ARMA processes we have in general $\bar{P}_m > \bar{P}_{m+1}$.

Since $\bar{P}_m$ is monotonically decreasing and also $\bar{P}_m > 0$, it follows that $\bar{P}_m$ will converge to a limit as m tends to infinity. A formal discussion of the convergence of $\bar{P}_m$ is given in Appendix B where it is also shown that

$$\bar{P}_\infty = \lim_{m\to\infty} \bar{P}_m = \lambda^2[E\{\phi(t)\psi^T(t)\}E\{\psi(t)\phi^T(t)\}]^{-1} \tag{18}$$

where $\psi(t)$ is the following infinite-dimensional vector:

$$\psi(t) = \frac{1}{C(q^{-1})}\begin{bmatrix} e(t-nc-1) \\ e(t-nc-2) \\ \vdots \\ \vdots \end{bmatrix}. \tag{19}$$

The limiting error covariance matrix $\bar{P}_\infty$ can be evaluted by solving a certain discrete Lyapunov equation (see (A.5) in Appendix A and (B.17) in Appendix B). Note that $\bar{P}_\infty$ is independent of $G(\cdot)$. We will show, however, in Section VI that the choice of $G(\cdot)$ affects the "convergence rate" of $\bar{P}_m$.

V. Comparison of the Accuracies of the Optimal IV Method and the Prediction Error Method

The prediction error method has been studied widely in the context of system identification [18], [19], [26]. The prediction error estimate of the parameters $\{a_i, c_i\}$ of an ARMA system is obtained by minimizing the loss function

$$V_N(\hat{a}_1, \cdots, \hat{a}_{na}, \hat{c}_1, \cdots, \hat{c}_{nc}) = \sum_{t=1}^{N} \epsilon^2(t) \tag{20}$$

where

$$\epsilon(t) = \frac{\hat{A}(q^{-1})}{\hat{C}(q^{-1})} y(t). \tag{21}$$

The prediction error estimate is known to be asymptotically normally distributed with the following normalized covariance matrix:

$$\lim_{N\to\infty} \frac{N}{\lambda^2} \operatorname{cov}\{\hat{a}_1, \cdots, \hat{a}_{na}, \hat{c}_1, \cdots, \hat{c}_{nc}\} = \left[E\left\{\begin{bmatrix} \psi_1(t) \\ -\psi_2(t)\end{bmatrix}[\psi_1^T(t), -\psi_2^T(t)]\right\}\right]^{-1}, \tag{22}$$

where

$$\psi_1^T(t) = \frac{1}{A(q^{-1})}[e(t-1), \cdots, e(t-na)], \tag{23}$$

$$\psi_2^T(t) = \frac{1}{C(q^{-1})}[e(t-1), \cdots, e(t-nc)]. \tag{24}$$

It is straightforward to show from (22) that the normalized covariance matrix of the AR parameter estimates obtained by the PEM is given by

$$P_{PEM} \triangleq \lim_{N\to\infty} \frac{N}{\lambda^2} \operatorname{cov}\{\hat{\theta}\} = [D_{11} - D_{12}D_{22}^{-1}D_{12}^T]^{-1}, \tag{25}$$

where

$$\hat{\theta} = [\hat{a}_1, \cdots, \hat{a}_{na}]^T, \tag{26a}$$

$$D_{ij} = E\{\psi_i(t)\psi_j^T(t)\}, \quad i, j = 1, 2. \tag{26b}$$

The following result states that the optimal IV method has the same asymptotic accuracy as the PEM.

Theorem 3: Let $\bar{P}_\infty$ and P_{PEM} be the covariance matrices defined by (18) and (23)-(26), respectively. Then, under assumptions A1-A3 $\bar{P}_\infty = P_{PEM}$.

Proof: See Appendix A.

As was mentioned earlier, in the Gaussian case, the PEM is an efficient estimator, i.e., P_{PEM} equals the Cramér-Rao lower bound [19], [22]. We conclude therefore that the optimal IV method is an efficient estimator for Gaussian processes. If the data are not Gaussian, then the optimal IV estimate, like the PE estimate, will still give the minimum variance in the fairly large class of parameter estimators whose covariance matrices depend only on the second-order statistics of the data.

It is interesting to investigate the rate at which $\bar{P}_m$ converges to $\bar{P}_\infty = P_{PEM}$, since in practice the value of m cannot be too large. The "convergence rate" of $\bar{P}_m$ is illustrated by the following examples.

Example 1: Convergence of $\bar{P}_m$ to P_{PEM}: Consider the ARMA processes

$$S_1: (1-0.8q^{-1})y(t) = (1+0.7q^{-1})e(t)$$

$$S_2: (1-1.5q^{-1}+0.7q^{-2})y(t) = (1-q^{-1}+0.2q^{-2})e(t)$$

where in both cases $E\{e(t)e(s)\} = \delta_{t,s}(\lambda^2 = 1)$. For both S_1 and S_2 we evaluated P_{PEM} and the optimal covariance matrix $\bar{P}_m$, for $G(z) \equiv 1$ and $m = na, na+1, \cdots$. The results are shown in Table I, where $\bar{p}^{ij}_m$ denotes the (i,j)th element of $\bar{P}_m$. Note that $\bar{P}_m$ has essentially converged for $m = 15$. It is interesting to compare the accuracy of the optimal IV method to that of the basic modified Yule-Walker method ($m = na$, in which case the choice of Q is irrelevant). The difference in accuracies can be quite large. For example in the case of S_2, the ratio of the variances of $\hat{a}_1$ corresponding to the two methods is about 30. For higher order systems the difference of accuracy between the methods may be larger (see [30]).

Example 2: Convergence of $\bar{P}_m$ to P_{PEM}: Note that $\bar{P}_m$ approaches P_{PEM} more or less at an exponential rate (cf. Example 1). To investigate the convergence rate in more detail consider the general ARMA (1, 1) process

$$y(t) = -ay(t-1) + e(t) + ce(t-1). \tag{27}$$

Assuming that

$$\bar{P}_m = P_{PEM} + K\gamma^m, \qquad 0 \leq \gamma < 1, \quad K = \text{constant}, \tag{28}$$

it seems reasonable to plot $\ln[(\bar{P}_m - P_{PEM})/P_{PEM}]$ versus m. This is done in Figs. 1 and 2 for $G(q^{-1}) \equiv 1$ and different values of the parameters a and c. It can be seen that except for small values of m, the curves can be well approximated by straight lines. This justifies the assumption in (28). It is interesting to note that the convergence rate depends strongly on c, and only weakly on a. The convergence is particularly slow when c is close to -1 (zero near the unit circle).

Similar results hold for c close to $+1$. The large variations in convergence rates for different parameters of the data motivates the study of ways for improving the convergence rate. In the next section we show how the choice of $G(q^{-1})$ affects the convergence rate.

VI. The Optimal Choice of $G(q^{-1})$

In this section we show that the choice $G(q^{-1}) = 1/C^2(q^{-1})$ will ensure that optimal estimation accuracy is achieved for a finite m, in fact for $m = na$. To see this we state the following lemma. Note that in the following calculations we will add the subscript m to R, S, $z(t)$, etc., to emphasize their dependence on the number of instrumental variables.

Lemma 2: The matrices $\{\bar{P}_m\}$ form a nonincreasing sequence,

TABLE I
CONVERGENCE OF $\bar{P}_m$ TO $\bar{P}_\infty = P_{PEM}$ FOR S_1 AND S_2

	S_1	S_2		
m	$\bar{p}^{11}_m$	$\bar{p}^{11}_m$	$\bar{p}^{12}_m$	$\bar{p}^{22}_m$
1	0.471	--	--	--
2	0.426	52.190	-16.320	6.276
3	0.409	8.577	-2.181	1.689
4	0.401	3.298	-1.151	1.488
5	0.397	2.147	-1.093	1.485
6	0.394	1.804	-1.131	1.481
8	0.391	1.626	-1.189	1.461
10	0.390	1.589	-1.213	1.445
12	0.390	1.580	-1.222	1.436
14	0.389	1.577	-1.226	1.431
16	0.389	1.576	-1.227	1.429
18	0.389	1.576	-1.227	1.429
20	0.389	1.576	-1.227	1.429
p^{ij}_{PEM}	0.389	1.576	-1.227	1.429

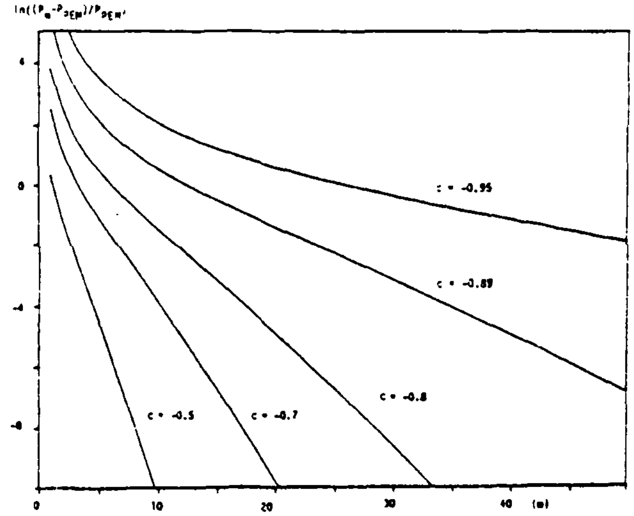

Fig. 1. The convergence of $\bar{P}_m$, $c = -0.9$, varying a.

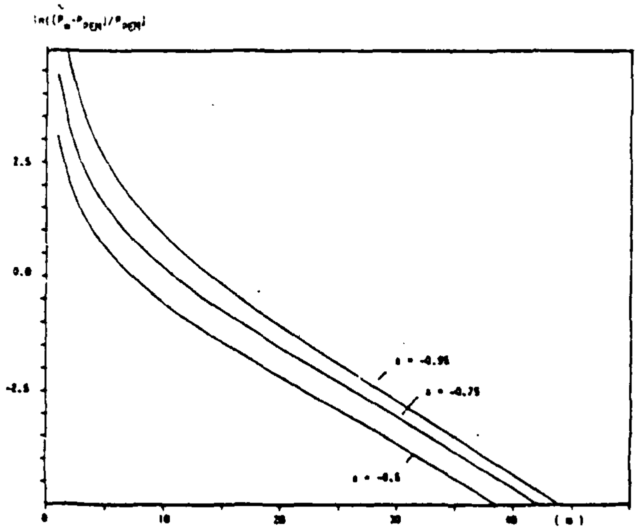

Fig. 2. The convergence of $\bar{P}_m$, $c = -0.9$, varying a.

i.e., $\bar{P}_{na} \geq \bar{P}_{na+1} \geq \cdots \geq \bar{P}_{\infty}$. Furthermore, all the equalities hold if and only if

$$R_m^T S_m^{-1} x_m = 0 \qquad \text{for } m \geq na \tag{29}$$

where R_m, S_m are as defined by (11), (15), and

$$x_m \triangleq E\left\{ C^2(q^{-1})G(q^{-1}) \begin{bmatrix} e(t-1) \\ \vdots \\ e(t-m) \end{bmatrix} \cdot \frac{C^2(q^{-1})G(q^{-1})}{A(q^{-1})} e(t-m-1) \right\}. \tag{30}$$

Proof: See Appendix C. ∎

It is now easy to see that the choice $G(q^{-1}) = 1/C^2(q^{-1})$, will satisfy (29) and is, therefore, optimal (although not necessarily the only optimal choice). We state this formally in the following theorem.

Theorem 4: Let assumptions A1–A3 hold true and consider the IV estimate (6) with $m = na$ and $G(q^{-1}) = 1/C^2(q^{-1})$ (the choice of Q is irrelevant in this case). Under these conditions the IV estimate will be optimal in the sense that its asymptotic ($N \to \infty$) covariance matrix equals $\bar{P}_\infty (= P_{PEM})$.

Proof: Direct consequence of Lemma 2. ∎

VII. Conclusions

We presented a detailed analysis of the accuracy aspects of a general IV method for estimating the AR parameters of an ARMA process. The basic accuracy result (Theorem 1) is useful for evaluating the performance bounds for the various MYW related estimation techniques discussed in the literature. See, for example, the discussion in [23], [30].

More importantly, Theorem 1 can be used to investigate the existence of optimal IV methods. We derived a lower bound on the estimation accuracy of IV estimators and presented methods for achieving this bound.

The first method involved an optimal weighting matrix $Q = S^{-1}$, and letting the number m of instrumental variables increase to infinity. In this case the choice of the filter $G(q^{-1})$ becomes unimportant and we may set $G(q^{-1}) = 1$ (see Theorem 2).

The second method involved an optimal filtering operation $G(q^{-1}) = 1/C^2(q^{-1})$. In this case the asymptotic bound is achieved for $m = na$, and the choice of the weigting matrix Q is unimportant (see Theorem 4).

Furthermore, we have shown that the optimal IV method has the same (asymptotic) accuracy as the prediction error method (see Theorem 3).

The methods discussed above suggest two new algorithms for estimating the AR parameters of ARMA models. These algorithms are discussed in some detail in a companion paper [24]. Note that both of these methods require knowledge of certain quantities [such as $C(q^{-1})$] which are not available *a priori*. In [24] it is shown that replacing those quantities by their consistent estimates does not degrade the asymptotic estimation accuracy.

Finally, we note that the optimal weighting matrix $Q = S^{-1}$ (required by the first method) can be estimated without explicit estimation of the MA parameters. This is convenient in some applications where one needs only estimates of the AR parameters.

Appendix

Proof of Theorem 3

Let us introduce the following notation:

$$\tilde{r}_k = E\left\{ \frac{C(q^{-1})}{A(q^{-1})} e(t) \cdot \frac{1}{C(q^{-1})} e(t-k) \right\},$$

$$\tilde{R}_k = \begin{bmatrix} \tilde{r}_k \\ \tilde{r}_{k-1} \\ \vdots \\ \tilde{r}_{k-na+1} \end{bmatrix}. \tag{A.1}$$

It is straightforward to show that

$$\tilde{r}_k + a_1 \tilde{r}_{k-1} + \cdots + a_{na} \tilde{r}_{k-na} = E\left\{ C(q^{-1}) e(t) \cdot \frac{1}{C(q^{-1})} e(t-k) \right\} = 0, \quad \text{for } k \geq nc+1, \tag{A.2}$$

and hence

$$\tilde{R}_k = A \tilde{R}_{k-1}, \qquad \text{for } k \geq nc+1 \tag{A.3}$$

where A is the following companion matrix associated with the polynomial $A(z)$:

$$A = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{na} \\ 1 & & 0 & \vdots \\ & \ddots & & \\ 0 & & 1 & 0 \end{bmatrix}. \tag{A.4}$$

It follows from (A.1)–(A.3) and (18) that

$$\bar{P}_\infty^{-1} - A \bar{P}_\infty^{-1} A^T = \frac{1}{\lambda^2} \left\{ \sum_{i=nc}^{\infty} \tilde{R}_i \tilde{R}_i^T - A \left[\sum_{i=nc}^{\infty} \tilde{R}_i \tilde{R}_i^T \right] A^T \right\} = \frac{1}{\lambda^2} \tilde{R}_{nc} \tilde{R}_{nc}^T.$$

In other words, $\bar{P}_\infty^{-1}$ satisfies the following Lyapunov equation [see also (B.17)]

$$\bar{P}_\infty^{-1} - A \bar{P}_\infty^{-1} A^T = \frac{1}{\lambda^2} \tilde{R}_{nc} \tilde{R}_{nc}^T. \tag{A.5}$$

Since A is a stability matrix, (A.5) has a *unique* solution, (see, e.g., [20]). To show that $\bar{P}_\infty = P_{PEM}$ it is thus sufficient and necessary to show that P_{PEM}^{-1} satisfies the same Lyapunov equation (A.5). We do this in the following steps. First note that

$$A \psi_1(t) = \psi_1(t+1) - e(t) u_1 \tag{A.6}$$

where

$$u_1 = [\underbrace{1 \; 0 \; \cdots \; 0}_{na}]^T \tag{A.7}$$

and therefore

$$A D_{11} A^T = D_{11} - \lambda^2 u_1 u_1^T \tag{A.8}$$

where D_{11} is as defined in (26b).

Next, we introduce

$$C = \begin{bmatrix} -c_1 & \cdots & -c_{nc} \\ 1 & 0 & 0 \\ & \ddots & \vdots \\ 0 & 1 & 0 \end{bmatrix} \tag{A.9}$$

and note that since $c_{nc} \neq 0$ the companion matrix C is nonsingular, and that

$$C \psi_2(t) = \psi_2(t+1) - e(t) u_2 \tag{A.10}$$

where

$$u_2 = [\underbrace{1\ 0\ \cdots\ 0}_{nc}]^T. \tag{A.11}$$

We can now write

$$AD_{12}D_{22}^{-1}D_{12}^TA^T - AD_{12}C^T(CD_{22}C^T)^{-1}CD_{12}^TA^T$$

$$= (D_{12} - \lambda^2 u_1 u_2^T)(D_{22} - \lambda^2 u_2 u_2^T)^{-1}(D_{12}^T - \lambda^2 u_2 u_1^T) \tag{A.12}$$

where D_{12}, D_{21}, D_{22} are as defined in (26b). It follows from the matrix inversion lemma that

$$(D_{22} - \lambda^2 u_2 u_2^T)^{-1} = D_{22}^{-1} + \frac{\lambda^2 D_{22}^{-1} u_2 u_2^T D_{22}^{-1}}{1 - \lambda^2 u_2^T D_{22}^{-1} u_2}. \tag{A.13}$$

By using (A.8), (A.12), and (A.13) we obtain after some straightforward but somewhat tedious calculations

$$P_{\text{PEM}}^{-1} - AP_{\text{PEM}}^{-1}A^T = \lambda^2 \frac{(u_1 - D_{12}D_{22}^{-1}u_2)(u_1 - D_{12}D_{22}^{-1}u_2)^T}{1 - \lambda^2 u_2^T D_{22}^{-1} u_2}. \tag{A.14}$$

According to a well-known formula for the inverse of the covariance matrix of an AR process [27], we have

$$D_{22}^{-1}u_2 = \frac{1}{\lambda^2}\left\{\begin{bmatrix} 1 \\ c_1 \\ \vdots \\ c_{nc-1} \end{bmatrix} - c_{nc}\begin{bmatrix} c_{nc} \\ c_{nc-1} \\ \vdots \\ c_1 \end{bmatrix}\right\}. \tag{A.15}$$

To proceed we note the following properties of the covariance elements of D_{12}. Let

$$\gamma_k = E\left\{\frac{1}{A(q^{-1})}e(t) \cdot \frac{1}{C(q^{-1})}e(t-k)\right\}. \tag{A.16}$$

We have

$$\gamma_k + c_1\gamma_{k-1} + \cdots + c_{nc}\gamma_{k-nc}$$

$$= E\left\{\frac{C(q^{-1})}{A(q^{-1})}e(t) \cdot \frac{1}{C(q^{-1})}e(t-k)\right\}$$

$$= \bar{r}_k, \quad \text{for all } k, \tag{A.17}$$

and

$$\gamma_k + c_1\gamma_{k+1} + \cdots + c_{nc}\gamma_{k+nc}$$

$$= E\left\{\frac{1}{A(q^{-1})}e(t) \cdot e(t-k)\right\}$$

$$= \begin{cases} \lambda^2 & k = 0 \\ 0 & k < 0. \end{cases} \tag{A.18}$$

If similarly to (A.1) we introduce

$$\Gamma_k = \begin{bmatrix} \gamma_k \\ \gamma_{k-1} \\ \vdots \\ \gamma_{k-na+1} \end{bmatrix}.$$

then from (A.15)–(A.18) we have that

$$D_{12}D_{22}^{-1}u_2 = [\Gamma_0\ \Gamma_1\ \cdots\ \Gamma_{nc-1}]\left\{\begin{bmatrix} 1 \\ c_1 \\ \vdots \\ c_{nc-1} \end{bmatrix} - c_{nc}\begin{bmatrix} c_{nc} \\ c_{nc-1} \\ \vdots \\ c_1 \end{bmatrix}\right\}\Big/\lambda^2$$

$$= [\lambda^2 u_1 - c_{nc}\Gamma_{nc} - c_{nc}(\bar{R}_{nc} - \Gamma_{nc})]/\lambda^2$$

$$= u_1 - \frac{c_{nc}}{\lambda^2}\bar{R}_{nc}$$

which gives

$$u_1 - D_{12}D_{22}^{-1}u_2 = \frac{c_{nc}}{\lambda^2}\bar{R}_{nc}.$$

To evaluate the denominator of the right-hand side of (A.14), we use (A.15) to obtain

$$1 - \lambda^2 u_2^T D_{22}^{-1} u_2 = 1 - (1 - c_{nc}^2) = c_{nc}^2.$$

It follows that the right-hand side of (A.14) reduces to $1/\lambda^2 \bar{R}_{nc}\bar{R}_{nc}^T$, which is precisely the right-hand side of (A.5). We have shown that $\bar{P}_\infty^{-1}$ and $\bar{P}_{\text{PEM}}^{-1}$ obey the same Lyapunov equation and therefore $\bar{P}_\infty^{-1} = \bar{P}_{\text{PEM}}^{-1}$.

Appendix B

Convergence of $\bar{P}_m$

In this Appendix we consider the convergence as $m \to \infty$ of the inverse of the optimal error covariance matrix

$$\bar{P}_m^{-1} = R_m^T S_m^{-1} R_m \tag{B.1}$$

where R_m and S_m are defined by (11) and (15), respectively. We start by introducing the following notation:

$$\bar{r}_k = E\{y(t) \cdot G(q^{-1})y(t-k)\},$$

$$\bar{R}_k = \begin{bmatrix} \bar{r}_k \\ \bar{r}_{k-1} \\ \vdots \\ \bar{r}_{k-na+1} \end{bmatrix}. \tag{B.2}$$

Note that

$$\bar{r}_k + a_1\bar{r}_{k-1} + \cdots + a_{na}\bar{r}_{k-na}$$

$$= E\{A(q^{-1})y(t) \cdot G(q^{-1})y(t-k)\} = 0, \qquad k \geq nc+1. \tag{B.3}$$

If we let A be the companion matrix defined in (A.4), then (B.3) implies that

$$\bar{R}_k = A\bar{R}_{k-1} \qquad k \geq nc+1. \tag{B.4}$$

Let us also introduce

$$\tilde{y}(t) = C(q^{-1})G(q^{-1})y(t),$$

$$\psi_m^T = E\{\tilde{y}(t) \cdot [\tilde{y}(t-1)\ \cdots\ \tilde{y}(t-m)]\},$$

$$\alpha_m^2 = E\{\tilde{y}^2(t)\} - \psi_m^T S_m^{-1}\psi_m. \tag{B.5}$$

We can now state the following result.

Lemma B1: Consider the sequence of matrices $\bar{P}_m^{-1}$, $m = 1, 2, \cdots$ defined by (B.1). The following Lyapunov-type equation

holds true

$$\bar{P}_{m+1}^{-1} - A\bar{P}_m^{-1}A^T = \frac{1}{\alpha_m^2}(\bar{R}_{nc} - AR_m^T S_m^{-1}\psi_m) \cdot (\bar{R}_{nc} - AR_m^T S_m^{-1}\psi_m)^T, \quad m = 1, 2, \cdots. \tag{B.6}$$

Proof: First note that according to (B.4)

$$R_{m+1}^T = E\{\phi(t) \cdot z_{m+1}^T(t)\} = [\bar{R}_{nc}, \bar{R}_{nc+1}, \cdots, \bar{R}_{nc+m}] = [\bar{R}_{nc}, AR_m^T].$$

Next, we have

$$S_{m+1}^{-1} = \begin{bmatrix} E\{y^2(t)\} & \psi_m^T \\ \psi_m & S_m \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & S_m^{-1} \end{bmatrix} + \frac{1}{\alpha_m^2}\begin{bmatrix} -1 \\ S_m^{-1}\psi_m \end{bmatrix}[-1, \psi_m^T S_m^{-1}].$$

Therefore, we can write

$$\bar{P}_{m+1}^{-1} = R_{m+1}^T S_{m+1}^{-1} R_{m+1} = AR_m^T S_m^{-1} R_m A^T + \frac{1}{\alpha_m^2}(\bar{R}_{nc} - AR_m^T S_m^{-1}\psi_m)(\bar{R}_{nc} - AR_m^T S_m^{-1}\psi_m)^T$$

which concludes the proof. ▨

Next, we study the limit as $m \to \infty$ of the right-hand side of (B.6).

Lemma B2: Let $m \to \infty$. Then, under assumptions A1-A3

$$\alpha_m^2 \to \lambda^2 \tag{B.7a}$$

$$\bar{R}_{nc} - AR_m^T S_m^{-1}\psi_m \to E\left\{\phi(t) \cdot \frac{1}{C(q^{-1})} e(t - nc - 1)\right\} \triangleq \tilde{R}_{nc}. \tag{B.7b}$$

Proof: Define

$$\sum_{i=0}^{\infty} h_i q^{-i} \triangleq \frac{A(q^{-1})}{C^2(q^{-1})G(q^{-1})}, \quad (h_0 = 1). \tag{B.8}$$

Due to assumptions A1 and A3

$$|h_k| < c\mu^k \tag{B.9}$$

where c is a constant, and $0 < \mu < 1$ is the maximum modulus of the zeros of $C^2(q^{-1})G(q^{-1})$. Now $A(q^{-1})y(t) = C^2(q^{-1})G(q^{-1})e(t)$, so that for m large enough we can write

$$y(t) + h_1 y(t-1) + \cdots + h_m y(t-m) + 0(\mu^{m+1}) = e(t). \tag{B.10}$$

It follows from (B.10) that

$$\psi_m = -S_m \begin{bmatrix} h_1 \\ \vdots \\ h_m \end{bmatrix} + 0(\mu^{m+1}). \tag{B.11}$$

Hence

$$AR_m^T S_m^{-1}\psi_m = -AR_m^T \begin{bmatrix} h_1 \\ \vdots \\ h_m \end{bmatrix} + 0(m\mu^{m+1}), \tag{B.12}$$

and

$$\psi_m^T S_m^{-1}\psi_m = -\psi_m^T \begin{bmatrix} h_1 \\ \vdots \\ h_m \end{bmatrix} + 0(m\mu^{m+1}). \tag{B.13}$$

Consider first (B.12). We have

$$\begin{aligned} R_m^T \begin{bmatrix} h_1 \\ \vdots \\ h_m \end{bmatrix} &= E\left\{\phi(t) \cdot G(q^{-1})\left[\sum_{i=1}^{m} h_i y(t - nc - i)\right]\right\} \\ &= E\left\{\phi(t) \cdot G(q^{-1})\left[\sum_{i=0}^{\infty} h_i y(t - nc - i) - y(t - nc) + 0(\mu^{m+1})\right]\right\} \\ &= E\left\{\phi(t) \cdot \frac{1}{C(q^{-1})} e(t - nc)\right\} - E\{\phi(t) \cdot G(q^{-1})y(t - nc)\} + 0(\mu^{m+1}). \end{aligned} \tag{B.14}$$

Further straightforward calculations give

$$A\phi(t) = \phi(t+1) - C(q^{-1})e(t)u_1, \quad u_1 = \left.\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right\} na. \tag{B.15}$$

Combining (B.14) and (B.15), we obtain

$$\begin{aligned} AR_m^T \begin{bmatrix} h_1 \\ \vdots \\ h_m \end{bmatrix} &= \tilde{R}_{nc} - c_{nc}\lambda^2 u_1 - \bar{R}_{nc} + c_{nc}\lambda^2 u_1 + 0(\mu^{m+1}) \\ &= \tilde{R}_{nc} - \bar{R}_{nc} + 0(\mu^{m+1}). \end{aligned} \tag{B.16}$$

This equation together with (B.12) implies (B.7b). Next consider (B.13). We have

$$\begin{aligned} \psi_m^T \begin{bmatrix} h_1 \\ \vdots \\ h_m \end{bmatrix} &= E\left\{y(t) \cdot \left[\sum_{i=1}^{m} h_i y(t-i)\right]\right\} \\ &= E\left\{y(t) \cdot \left[\sum_{i=0}^{\infty} h_i y(t-i) - y(t) + 0(\mu^{m+1})\right]\right\} \\ &= E\{y(t) \cdot e(t)\} - E\{y^2(t)\} + 0(\mu^{m+1}) \\ &= \lambda^2 - E\{y^2(t)\} + 0(\mu^{m+1}) \end{aligned}$$

which together with (B.5) and (B.13) proves (B.7a). This concludes the proof of Lemma B2.

It is now straightforward to evaluate $\bar{P}_\infty \triangleq \lim_{m\to\infty} \bar{P}_m$. The limit exists since we have shown earlier that $\bar{P}_m \geq \bar{P}_{m+1} > 0$ (see Lemma 2). Furthermore, it follows from Lemmas B1 and B2 that $\bar{P}_\infty^{-1}$ satisfies

$$\bar{P}_\infty^{-1} - A\bar{P}_\infty^{-1}A^T = \frac{1}{\lambda^2}\tilde{R}_{nc}\tilde{R}_{nc}^T. \tag{B.17}$$

As is well known, under the given assumptions the solution of (B.17) is unique and is given by

$$\bar{P}_\infty^{-1} = \frac{1}{\lambda^2} \sum_{k=0}^{\infty} A^k \bar{R}_{nc} \bar{R}_{nc}^T (A^T)^k.$$

In Appendix A we have shown that $\bar{R}_k = A\bar{R}_{k-1}$. Therefore,

$$\bar{P}_\infty = \lambda^2 \left\{ [\bar{R}_{nc} \bar{R}_{nc+1} \cdots] \begin{bmatrix} \bar{R}_{nc}^T \\ \bar{R}_{nc+1}^T \\ \vdots \\ \vdots \end{bmatrix} \right\}^{-1}$$

which is precisely (18).

Appendix C

Proof of Lemma 2

Note that we can write

$$S_{m+1} = \begin{bmatrix} S_m & \psi_m \\ \psi_m^T & \sigma \end{bmatrix},$$

$$\psi_m = E\{C(q^{-1})z_m(t) \cdot C(q^{-1})G(q^{-1})y(t-nc-m-1)\},$$

$$\sigma = E\{C(q^{-1})G(q^{-1})y(t)\}^2,$$

and

$$R_{m+1} = \begin{bmatrix} R_m \\ \phi_m^T \end{bmatrix},$$

$$\phi_m = E\{\phi(t) \cdot G(q^{-1})y(t-nc-m-1)\}.$$

Therefore,

$$\bar{P}_{m+1}^{-1} = R_{m+1}^T S_{m+1}^{-1} R_{m+1} = [R_m^T \phi_m] \cdot \left\{ \begin{bmatrix} I \\ 0 \end{bmatrix} S_m^{-1} [I \quad 0] + \alpha_m \begin{bmatrix} S_m^{-1}\psi_m \\ -1 \end{bmatrix} [\psi_m^T S_m^{-1}, \ -1] \right\} \begin{bmatrix} R_m \\ \phi_m^T \end{bmatrix}$$

$$= \bar{P}_m^{-1} + \alpha_m [\phi_m - R_m^T S_m^{-1} \psi_m][\phi_m - R_m^T S_m^{-1} \psi_m]^T \qquad \text{(C.1)}$$

where

$$\alpha_m = \frac{1}{\sigma - \psi_m^T S_m^{-1} \psi_m}.$$

Since $S_m > 0$ for all m, we have $0 < \alpha_m < \infty$. This proves the order relation $\bar{P}_m \geq \bar{P}_{m+1}$. Furthermore, it follows from (C.1) that

$$\{\bar{P}_m = \bar{P}_{m+1},\ m \geq na\} \Leftrightarrow \{\phi_m = R_m^T S_m^{-1} \psi_m,\ m \geq na\}. \qquad \text{(C.2)}$$

To obtain (29) from (C.2) notice first that

$$x_m = E\left\{ \frac{C^2(q^{-1})G(q^{-1})}{A(q^{-1})} \begin{bmatrix} e(t-1) \\ \vdots \\ e(t-m) \end{bmatrix} \cdot A(q) \frac{C^2(q^{-1})G(q^{-1})}{A(q^{-1})} e(t-m-1) \right\}$$

$$= E\left\{ C(q^{-1})G(q^{-1}) \begin{bmatrix} y(t-1) \\ \vdots \\ y(t-m) \end{bmatrix} \cdot \{C(q^{-1})G(q^{-1})[y(t-1) \cdots y(t-m)]a + C(q^{-1})G(q^{-1})y(t-m-1)\} \right\} = S_m a + \psi_m \qquad \text{(C.3)}$$

where

$$a = [0, \ \cdots, \ 0, \ a_{na}, \ \cdots, \ a_1]^T.$$

Next, introduce

$$\bar{r}_k = E\{y(t) \cdot G(q^{-1})y(t-k)\}; \ \bar{R}_k = \begin{bmatrix} \bar{r}_k \\ \bar{r}_{k-1} \\ \vdots \\ \bar{r}_{k-na+1} \end{bmatrix} \qquad \text{(C.4)}$$

and note that

$$\bar{r}_k + a_1\bar{r}_{k-1} + \cdots + a_{na}\bar{r}_{k-na} = E\{C(q^{-1})e(t)G(q^{-1})y(t-k)\} = 0, \ k \geq nc+1 \qquad \text{(C.5)}$$

and, therefore, that

$$\bar{R}_k + a_1\bar{R}_{k-1} + \cdots + a_{na}\bar{R}_{k-na} = 0, \quad \text{for } k \geq nc + na.$$

It follows from (C.3) that

$$\phi_m - R_m^T S_m^{-1} \psi_m = \phi_m + R_m^T a - R_m^T S_m^{-1} x_m.$$

However, from (C.1) and (C.2) we have,

$$\phi_m + R_m^T a = \bar{R}_{nc+m} + [\bar{R}_{nc} \cdots \bar{R}_{nc+m-1}] \begin{bmatrix} 0 \\ \vdots \\ 0 \\ a_{na} \\ \vdots \\ a_1 \end{bmatrix}$$

$$= \bar{R}_{nc+m} + a_1\bar{R}_{nc+m-1} + \cdots + a_{na}\bar{R}_{nc+m-na} = 0, \text{ for } m \geq na.$$

Hence, (C.2) reduces to (29) and the proof is completed.

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APPENDIX C

LEAST-SQUARES, YULE-WALKER, AND OVERDETERMINED YULE-WALKER ESTIMATION OF AR PARAMETERS: A MONTE CARLO ANALYSIS OF FINITE-SAMPLE PROPERTIES

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LEAST-SQUARES, YULE-WALKER, AND OVERDETERMINED YULE-WALKER ESTIMATION OF AR PARAMETERS: A MONTE CARLO ANALYSIS OF FINITE-SAMPLE PROPERTIES

Petre Stoica, Benjamin Friedlander, and Torsten Söderström

ABSTRACT

A Monte Carlo analysis of the accuracy properties of least squares (LS), Yule-Walker (YW), and the overdetermined Yule-Walker (OYW) methods for estimating the parameters of autoregressive (AR) processes is presented. Comparisons of the estimated finite-sample accuracy to the theoretical asymptotic accuracy are included. It is shown that considerable differences may occur in some cases. Choice of the number of equations in the YW system of equations is discussed. Some remarks concerning the feasibility and usefulness of an analytical study of the finite-sample accuracy properties are also included.

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1. INTRODUCTION

There are many parameter estimation methods in use today. For most of them, an asymptotic accuracy theory is available. The interest in establishing the accuracy properties of an estimation method is motivated by at least the following: (i) interval estimation; (ii) hypothesis testing; (iii) accuracy comparisons with other estimation methods; (iv) accuracy optimization with respect to some "design variables" which are at the disposal of the user. The asymptotic accuracy theory has often been used for solving problems such as those listed above. However, in some cases, the asymptotic theory is not applicable for the sample lengths encountered in practice. In recent years, three main directions of research for overcoming this difficulty have appeared:

(i) Analytical studies aimed at establishing the exact finite-sample accuracy (moments or distribution) of the parameter estimators; this turned out to be possible in some simple cases (a typical example being the LS estimator of the first-order AR parameter). See [4, 6, 9, 10, 23].

(ii) Higher order approximations of the exact accuracy (moments or distribution). This approach proved more flexible than the one above, yet provided quite accurate approximations; see [1, 17-22].

(iii) Monte Carlo analysis of the finite-sample accuracy properties. This is a conceptually simple and general approach; see [5, 7, 8, 12, 15].

The aim of this paper is twofold: (i) To comment briefly on the three general approaches mentioned above. This general discussion is included in the next section. (ii) To consider a specific estimation problem for

illustrating some of the main issues addressed in the general discussion. More specifically, the problem of estimating the AR parameters is considered, and a Monte Carlo analysis of the accuracy properties of three methods frequently used for AR parameter estimation, Least Squares (LS), Yule-Walker (YW), and Overdetermined Yule-Walker (OYW), is presented.

It is perhaps worth remarking that most papers on small-sample properties have appeared in the econometric literature. A possible reason for this is the fact that econometricians deal more often than engineers with short samples (for example, containing around 50 data points). However, as the simulations of this paper will show, significant discrepancies between the finite sample behavior and that predicted by the asymptotic theory may well appear even for sample lengths encountered in engineering applications.

An outline of this paper is as follows. A general discussion on approaches to the analysis of finite-sample distributional properties of parameter estimators is given in the next section. In Section 3 we briefly describe the LS, YW, and OYW methods for estimating the AR parameters. Their asymptotic accuracy properties are reviewed in Section 4, where it is also shown that the asymptotic covariance matrix of the YW estimator is bound from above by the covariance matrix of the OYW estimator. Section 5 contains the results of a Monte Carlo analysis. Finally, some concluding remarks are presented in Section 6.

2. GENERAL DISCUSSION

There are at least two points which are of interest when discussing the approaches mentioned above: feasibility and usefulness.

For many estimators currently in use it is a formidable if not impossible task to establish the exact finite-sample properties of the distribution. In some simple cases, this task becomes feasible but the resulting exact expressions (for example, of the distribution moments) are so complicated that their usefulness may be questioned (see [10] and its references where a cumbersome formula is given for the finite-sample variance of the estimated parameter of a first-order AR process).

Specifically, let us suppose that θ is the unknown parameter vector and $\hat{\theta}_N$ its estimate obtained from an N-length sample. Introduce the normalized covariance matrix of the estimation errors

$$P_N(\theta) = N\, E\left\{(\hat{\theta}_N-\theta)(\hat{\theta}_N - \theta)^T\right\} , \tag{1}$$

and let $P_\infty(\theta)$ denote the asymptotic covariance matrix

$$P_\infty(\theta) = \lim_{N \to \infty} P_N(\theta) . \tag{2}$$

For many (consistent) estimators currently used in system identification, the above limit exists under weak conditions. Furthermore, we have

$$P_N(\theta) = P_\infty(\theta) + O(1/N^{1/2}) . \tag{3}$$

In practice, when using $P_N(\theta)$ or $P_\infty(\theta)$ for purposes such as interval estimation or hypothesis testing, we have to replace θ by $\hat{\theta}_N$. Since

$$\hat{\theta}_N - \theta = O(1/N^{1/2})$$

we have

$$P_N(\hat{\theta}_N) = P_N(\theta) + O(1/N^{1/2}) \quad .$$

On the other hand, from (3),

$$P_\infty(\hat{\theta}_N) = P_N(\theta) + O(1/N^{1/2}) \quad .$$

Thus, there is apparently no guarantee that $P_N(\hat{\theta}_N)$ is a better estimate of $P_N(\theta)$ than is $P_\infty(\hat{\theta}_N)$. The above discussion is valid for N sufficiently large. For "small" N, the above calculations are no longer valid. However, since $P_N(\theta)$ has a more complicated expression than $P_\infty(\theta)$, it may still be true that replacement of θ by $\hat{\theta}_N$ may in some cases lead to larger errors for $P_N(\cdot)$ than for $P_\infty(\cdot)$.

Next consider the problems of accuracy comparisons with other estimation methods, and accuracy optimization with respect to some "design variables" which are at the disposal of the user. For many estimation methods, there exist asymptotic results for both the optimization of accuracy and for comparison with the accuracy achieved by other estimation methods. However, these results may fail to apply for the sample lengths encountered in practice and are thus of little use in such situations. For example, asymptotically equivalent estimation methods have been shown to behave quite differently in the finite-sample case (see [5] and Section 5 of this paper). A considerable departure from asymptotic theory was reported in [15], where it was shown by extensive Monte Carlo simulations that in some cases the ordinary LS estimator may be better than the idealized Markov estimator in terms of both bias and variance. Since $P_N(\theta)$ will in general have a complicated expression, it is unlikely that analytical comparisons and optimizations of accuracy would be

possible in the finite-sample case. However, what should be possible is to evaluate $P_N(\theta)$ numerically for different N and θ. This may serve to identify sets in the parameter space and values of N for which one estimation method is better than another, and also to provide guidelines for "optimally" choosing the design variables defining the estimation method in question. The Monte Carlo analysis approach addresses the two objectives mentioned above. The Monte Carlo approach provides only an estimate of $P_N(\theta)$ (or of the distribution function). The larger the number of replications used in the Monte Carlo experiment, the better will be this estimate. Furthermore, a Monte Carlo analysis may be quite costly in terms of the computer time involved. However, when an expression for $P_N(\theta)$ is not available, the Monte Carlo analysis may be the only solution at hand. The Monte Carlo analysis may also be the preferred approach when the evaluation of the available expression for $P_N(\theta)$ requires a very cumbersome algorithm (see [3]). Extensive Monte Carlo analyses for evaluation of various instrumental variable methods are given in [27,28].

Finally, the development of higher order approximations for estimator accuracy seems to be the most promising one from a theoretical point of view. Essentially, it follows the lines of the asymptotic analysis but takes into account also some higher order terms in (3). Truncating asymptotic series expansions after a small number of terms is frequently used to get improved approximations of parameter estimate distribution or of its moments. A different approach to approximate analysis of finite-sample distribution was recently proposed in [20].

In many situations, the development of approximations is a more feasible theoretical approach than the development of exact formulas. Also, it should lead to more manageable expressions for the covariance matrix of the estimation errors, etc. We believe that this approach is a topic that warrants more attention. Some recent results on the finite-sample covariance structure of the sampled covariances of ARMA processes (see [2, 3]) might be useful in this context (at least for studying the so-called correlation-based techniques). We may also remark that Monte Carlo simulation results may be useful when deriving approximate finite-sample properties of the distribution by using the analytical approach of [20].

In the next section, we will consider three methods for estimating the AR parameters. Even if estimating the AR parameters is apparently one of the simplest dynamic estimation problems, an exact finite-sample accuracy theory does not seem to be available for any of the methods considered. An analysis of the finite sample properties is beyond the scope of this paper. Instead, we resort to Monte Carlo analysis to show that:

(i) The asymptotic and finite-sample accuracy properties may be quite different in some cases. (ii) The number of YW equations used for estimation has a considerable influence on the accuracy. (Some guidelines for choosing that number are discussed.) (iii) The LS method performs in most cases better than the other two methods tested.

3. ESTIMATION METHODS

Consider the following general AR process

$$y(t) + a_1 y(t-1) + \dots + a_n y(t-n) = e(t), \tag{4}$$

where $\{e(t)\}$ is a sequence of independent and identically distributed random variables with zero mean and variance denoted λ^2, and the real coefficients $\{a_i\}$ are such that the polynomial

$$A(z) = 1 + a_1 z + \ldots + a_n z^n, \tag{5}$$

has all its zeros outside the unit circle.

The AR model (4) is used in many applications in engineering, econometrics, biometrics, geophysics, etc. and a number of methods are available for estimating its parameters. Of these, perhaps the most commonly used ones are the following three.

3.1 The LS Method

Let θ denote the vector of unknown parameter

$$\theta = [a_1 \; \ldots \; a_n]^T \quad . \tag{6}$$

The LS estimate of θ is defined as

$$\hat{\theta} = \arg\min_{\theta} \sum_{t=n+1}^{N} [y(t) - \varphi^T(t)\,\theta]^2, \tag{7}$$

where

$$\varphi(t) = [-\,y(t-1) \; \ldots \; -\,y(t-n)]^T \tag{8}$$

After some straightforward calculations, (7) produces the result [29]

$$\hat{\theta}_{LS} = [\sum_{t=n+1}^{N} \varphi(t)\,\varphi^T(t)]^{-1} [\sum_{t=n+1}^{N} \varphi(t)\,y(t)\,] \quad . \tag{9}$$

The inverse in (9) exists at least for large N.

3.2 The YW Method

As can be easily seen from (4), the coefficients $\{a_i\}$ satisfy the following equations:

$$r_k + a_1 r_{k-1} + \dots + a_n r_{k-n} = 0, \quad k \geq 1, \tag{10}$$

where

$$r_k = E\{y(t)\, y(t+k)\},$$

and where $E\{\cdot\}$ denotes expectation.

Equations (10) are the so-called YW equations and the estimate obtained after replacing $\{r_k\}$ by

$$\hat{r}_k = \frac{1}{N} \sum_{t=1}^{N-k} y(t)\, y(t+k), \quad \hat{r}_{-k} = \hat{r}_k, \quad k = 1, 2, \dots, \tag{11}$$

in the first n equations of (10) is called the YW estimate. Thus, the YW estimate of θ is given by [27]

$$\begin{bmatrix} \hat{r}_0 & \cdots & \hat{r}_{n-1} \\ \vdots & \ddots & \vdots \\ \hat{r}_{n-1} & \cdots & \hat{r}_0 \end{bmatrix} \hat{\theta}_{YW} = - \begin{bmatrix} \hat{r}_1 \\ \vdots \\ \hat{r}_n \end{bmatrix} . \tag{12}$$

Numerically efficient algorithms for solving the linear system (12) exist. For example, the Levinson-Durbin algorithm solves (12) in $O(n^2)$ arithmetic

operations. The Toeplitz structure of the matrix in (12) makes the YW method more efficient numerically than the LS method (9). (Equation (9) needs approximately $n/2$ times more multiplications than (12).) The LS estimate (9) and the YW estimate (12) are, however, asymptotically equivalent. For large N we have

$$\hat{\theta}_{LS} = \hat{\theta}_{YW} + O\left(\frac{1}{N}\right) \quad . \tag{13}$$

This result can be readily established.

3.3 The Overdetermined YW Method

The OYW method is based on the recognition of the fact that the Yule-Walker equations (10) involving high lag coefficients $(r_k,\ k > n)$ should be considered when estimating the parameters $\{a_i\}$ of (4). Then, instead of (12), one obtains an overdetermined system of equations which is to be solved in a least-squares sense. The OYW estimate is thus given by

$$\left\| \begin{bmatrix} \hat{r}_0 & \cdots & \hat{r}_{n-1} \\ \vdots & & \vdots \\ \hat{r}_{m-1} & \cdots & \hat{r}_{m-n} \end{bmatrix} \hat{\theta}_{OYW} + \begin{bmatrix} \hat{r}_1 \\ \vdots \\ \hat{r}_m \end{bmatrix} \right\|_Q^2 = \min \tag{14}$$

where $\|x\|_Q^2 = x^T Q x$, and Q is a positive definite weighting matrix of dimension $m \times m$. A numerically stable procedure for solving (14) is the QR algorithm.

Intuitively, we expect that the additional equations in (14) will improve the estimation accuracy, unless the sequence of covariances r_k dies out rapidly. In other words, for narrowband processes (14) with a relatively

large m should be preferred to (12), while for broad-band processes (12) may be preferable. The choice of m is discussed in some more detail in Section 5. We generally set $Q = I$.

The above conjectures pertaining to the choice of m are supported by practical experience with the method (see, for example, the simulation results in Section 5). The practical experience contradicts once more the asymptotic theory. See the next section where it is shown that (12) is asymptotically more accurate than $\hat{\theta}_{OYW}$, (14), for any $m > n$.

4. ASYMPTOTIC DISTRIBUTIONS

The LS estimate $\hat{\theta}_{LS}$ is asymptotically normally distributed with mean equal to the true parameter vector θ and covariance matrix given by [27],

$$P = \frac{\lambda^2}{N} [E\{\varphi(t)\ \varphi^T(t)\}]^{-1} \quad . \tag{15}$$

In view of the equivalence (13), the YW estimate $\hat{\theta}_{YW}$ has the same asymptotic distribution.

It follows from [24] that the OYW estimate $\hat{\theta}_{OYW}$ is asymptotically normally distributed with mean θ and covariance matrix given by,

$$\overline{P} = \frac{\lambda^2}{N} (R^TQR)^{-1}R^TQSQR(R^TQR)^{-1}, \tag{16}$$

where

$$R = \begin{bmatrix} r_0 & \cdots & r_{n-1} \\ \vdots & & \vdots \\ r_{m-1} & \cdots & r_{m-n} \end{bmatrix} \quad ,$$

and

$$S = \begin{bmatrix} r_0 & \cdots & r_{m-1} \\ \vdots & \ddots & \vdots \\ r_{m-1} & \cdots & r_0 \end{bmatrix} .$$

The relation between the covariance matrices P and $\overline{P}$ is of interest. The following result holds.

Lemma. Consider the covariance matrices P and $\overline{P}$ defined by (15) and (16), respectively. Then,

$$\overline{P} \geq P \tag{17}$$

Proof. See the appendix.

The results in this section are valid for a "sufficiently large" N. What constitutes a sufficiently large N depends on the $\{a_i\}$ parameters, or more precisely, on the location of the zeros of the polynomial $A(z)$. This is illustrated in the next section.

5. MONTE CARLO ANALYSIS

In this section , we report the results obtained for the following two second-order AR processes:

$$S_1: (1 - 0.9q^{-1} + 0.2q^{-2})\, y(t) = e(t), \tag{18a}$$

and

$$S_2 : (1 - 1.75q^{-1} + 0.76q^{-2})\, y(t) = e(t) \quad . \tag{18b}$$

The poles of S_1 are located at 0.4 and 0.5; those of S_2 are equal to 0.8 and 0.95. For each system, 50 independent realizations of 2000 data points each have been generated. The noise sequence $\{e(t)\}$ was obtained using the pseudo-random number generator NORMAL included in the statistical library of the FELIX/IRIS computer. NORMAL generates independent normal variables with zero mean and unit variance. The initial values required to start the recurrent calculations in (17) and (18) were simply set to zero.

The first N samples of each realization, with $N = 100$, 300, 500, and 2000, have been used to estimate the system parameters. The LS, YW, and OYW methods briefly described in Section 3 have been used to get parameter estimates. The OYW method has been applied for various values of m (see (14)).

Let $\hat{a}_k^i$ denote the estimate of a_k obtained from the i-th data realization by using one of the three methods under consideration. The following quantities have been evaluated (for $k = 1, 2$).

$$\bar{a}_k = \frac{1}{50} \sum_{i=1}^{50} \hat{a}_k^i, \qquad \text{(mean value of } \hat{a}_k\text{)},$$

$$\delta(a_k) = \frac{\bar{a}_k - a_k}{a_k}, \qquad \text{(percentage bias of } \hat{a}_k\text{)},$$

$$\operatorname{var}(\hat{a}_k) = \frac{1}{50} \sum_{i=1}^{50} (\hat{a}_k^i - \bar{a}_k)^2, \qquad \text{(variance of } \hat{a}_k\text{)},$$

and

$$\mathrm{MSE}(\grave{a}_k) = \mathrm{var}(\hat{a}_k) + [\bar{a}_k - a_k]^2, \quad \text{(mean square error of } \grave{a}_k).$$

The results obtained in the different cases are displayed in Figures 1 through 6. The asymptotic values of $\mathrm{var}(a_k)$ are also shown (in Figures 3 and 6). (The same symbols are used for the Monte Carlo and asymptotic results. The asymptotic results are the ones not connected by straight lines.) For $\delta(\grave{a}_k)$, the asymptotic value is zero. The following remarks can be made regarding these results.

(1) For the LS method, asymptotic theory holds quite well for all the sample lengths considered, for both S_1 and for S_2. For the YW and OYW methods the situation is different. For S_1, asymptotic theory is applicable for reasonably short sample lengths (e.g., for N = 300). However, for S_2, a good agreement between finite-sample and asymptotic behavior was found only for very long sample lengths (N = 2000). For short sample lengths, considerable differences between asymptotic theory and practical behavior occurred, especially for the YW method. For sample lengths of 100, 300, and 500, the YW method is by far the least accurate of those tested, despite the fact that the asymptotic theory recommends it as being the best. For the OYW method with $m = 20$, 30, or 40, the differences beteen asymptotic theory and practical results are not so large as for the YW method (e.g., for $m = 20$ and 30, the estimated and asymptotic values of the variances are in agreement for $N \geq 300$). It is interesting to note that for large m (e.g. $m = 40$), the finite-sample variances may be smaller than the corresponding asymptotic values.

(2) The LS method outperforms the YW and OYW methods. It gave the smallest MSE's in almost all the experiments performed. In most cases the LS method is superior to the YW and OYW methods in terms of both bias and variance of the parameter estimates. The superiority of the LS method over the YW and OYW methods is clear in the case of S_2. For S_1, the LS method and the YW method gave quite similar results.

The ranking of the OYW methods $(m \geq na)$ appears to be in accordance with the asymptotic theory only for S_1. For this system, $m = 2$ (corresponding to the YW method) gave the best results; when m was increased beyond 2, the estimation accuracy deteriorated. For S_2, the choice of m to get "best" accuracy is no longer so clear. Here, the "optimal" finite-sample value of m is certainly larger than the asymptotically optimal value $m = 2$. This was also the conclusion of a large number of empirical studies reported in the signal processing literature. It is difficult, however, to give precise rules for choosing m. In loose terms, the closer the system poles are to the unit circle, the larger should be m. For a given system, the "optimal" value of m depends on N. The larger N the smaller should be m (see, for example, Figure 4).

6. CONCLUSIONS

We presented a Monte Carlo analysis of the accuracy properties of several methods for estimating the parameters of an autoregressive process. The differences between finite-sample accuracy and the theoretical asymptotic accuracy were discussed. These results provide some useful insights into the behavior of these estimators.

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APPENDIX: PROOF OF THE LEMMA

Let

$$\tilde{P} = \frac{\lambda^2}{N} (R^T S^{-1} R)^{-1} \tag{A.1}$$

It is straightforward to show that

$$\bar{P} - \tilde{P} = \frac{\lambda^2}{N} \left[(R^T Q R)^{-1} R^T Q - (R^T S^{-1} R)^{-1} R^T S^{-1} \right] S \, [(R^T Q R)^{-1} R^T Q - (R^T S^{-1} R)^{-1} R^T S^{-1}]^T \tag{A.2}$$

It follows that $\bar{P} \geq \tilde{P}$.

To conclude the proof, we next show that $\tilde{P} \geq P$. This is equivalent to showing that

$$E\{\varphi(t)\,\varphi^T(t)\} - R^T S^{-1} R \geq 0$$

which in turn is equivalent to

$$\left[\begin{array}{c|c} E\{\varphi(t)\,\varphi^T(t)\} & R^T \\ \hline R & S \end{array}\right] = E\left\{ \left[\begin{array}{c} y(t-1) \\ \vdots \\ y(t-n) \\ \hline y(t-1) \\ \vdots \\ y(t-m) \end{array}\right] [y(t-1), \ldots\, y(t-n)\; y(t-n), \ldots, y(t-m)] \right\} \geq 0$$

$\Delta\ \Delta\ \Delta$

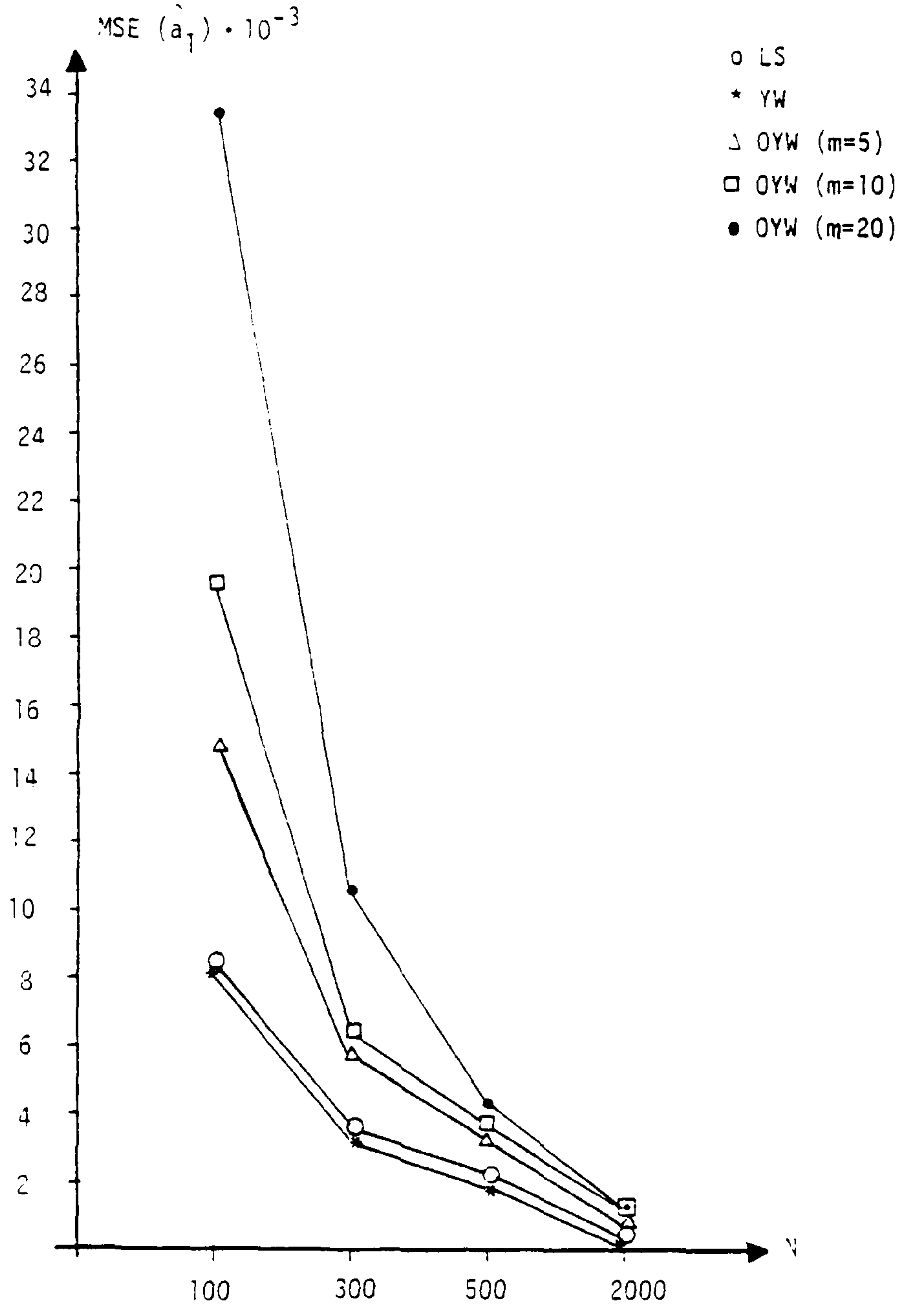

Figure 1A: Mean Square Error of $\hat{a}_1$ for S_1

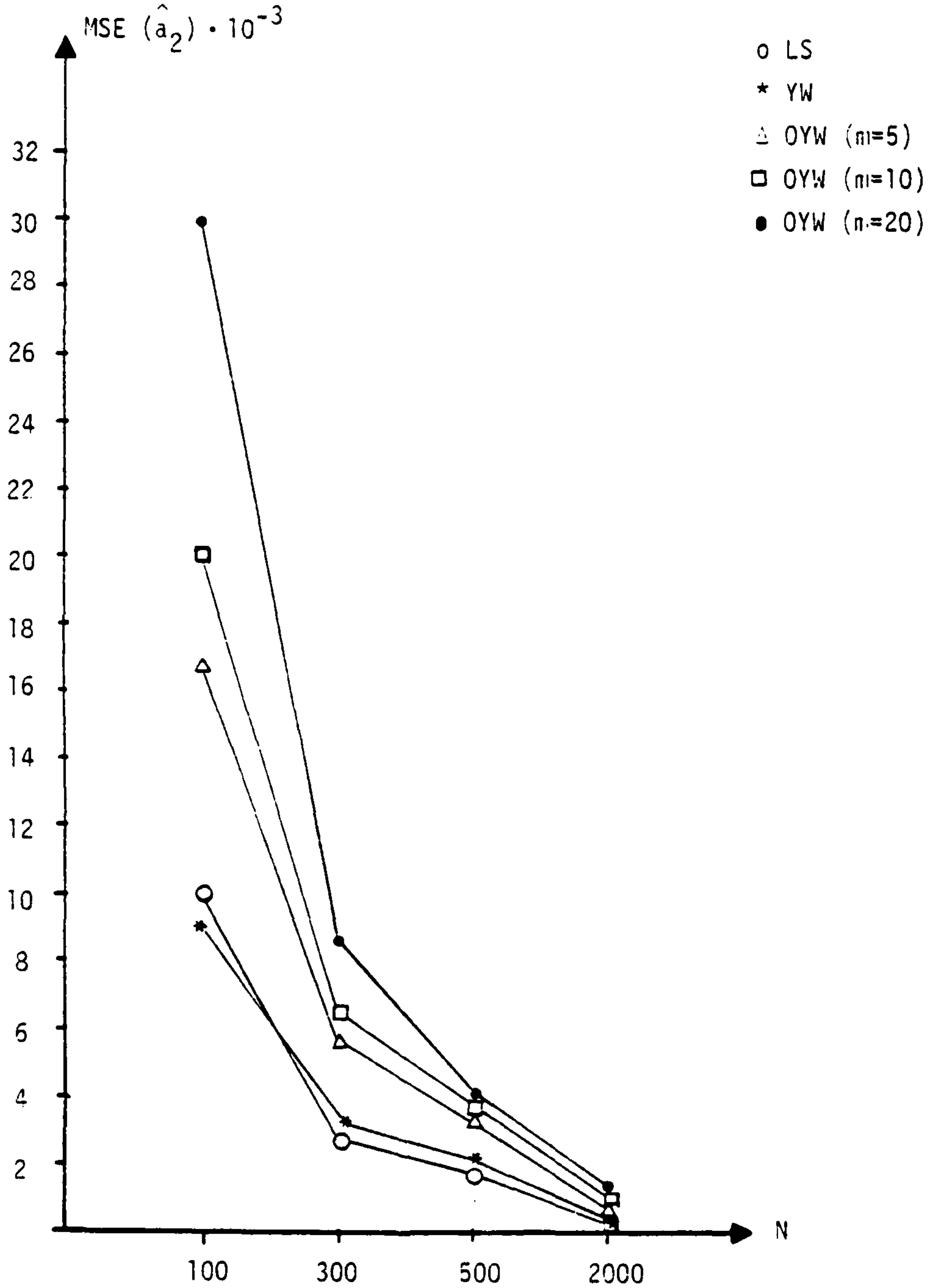

Figure 1B: Mean Square Error of $\hat{a}_2$ for S_1

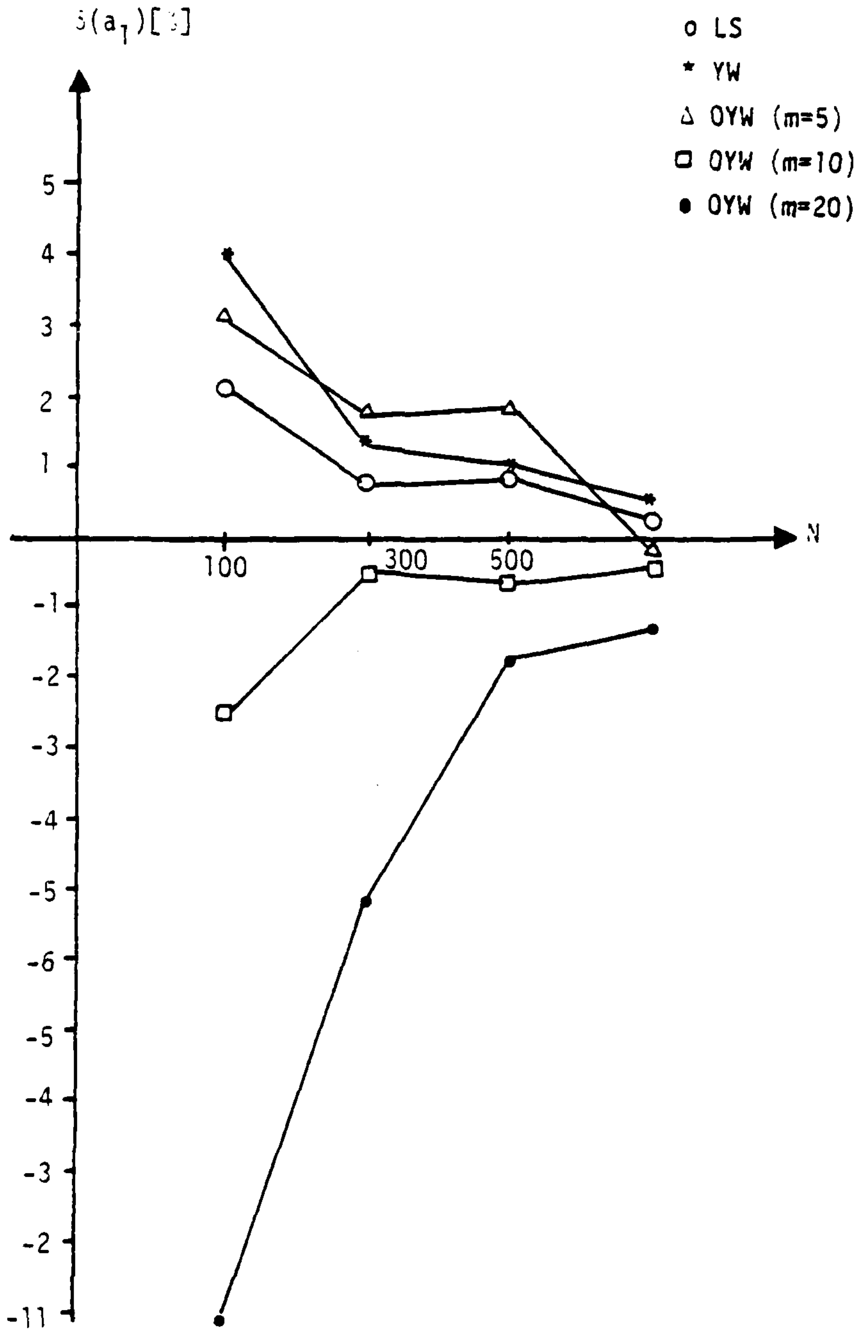

Figure 2A: Percentage Bias of $\hat{a}_1$ for S_1

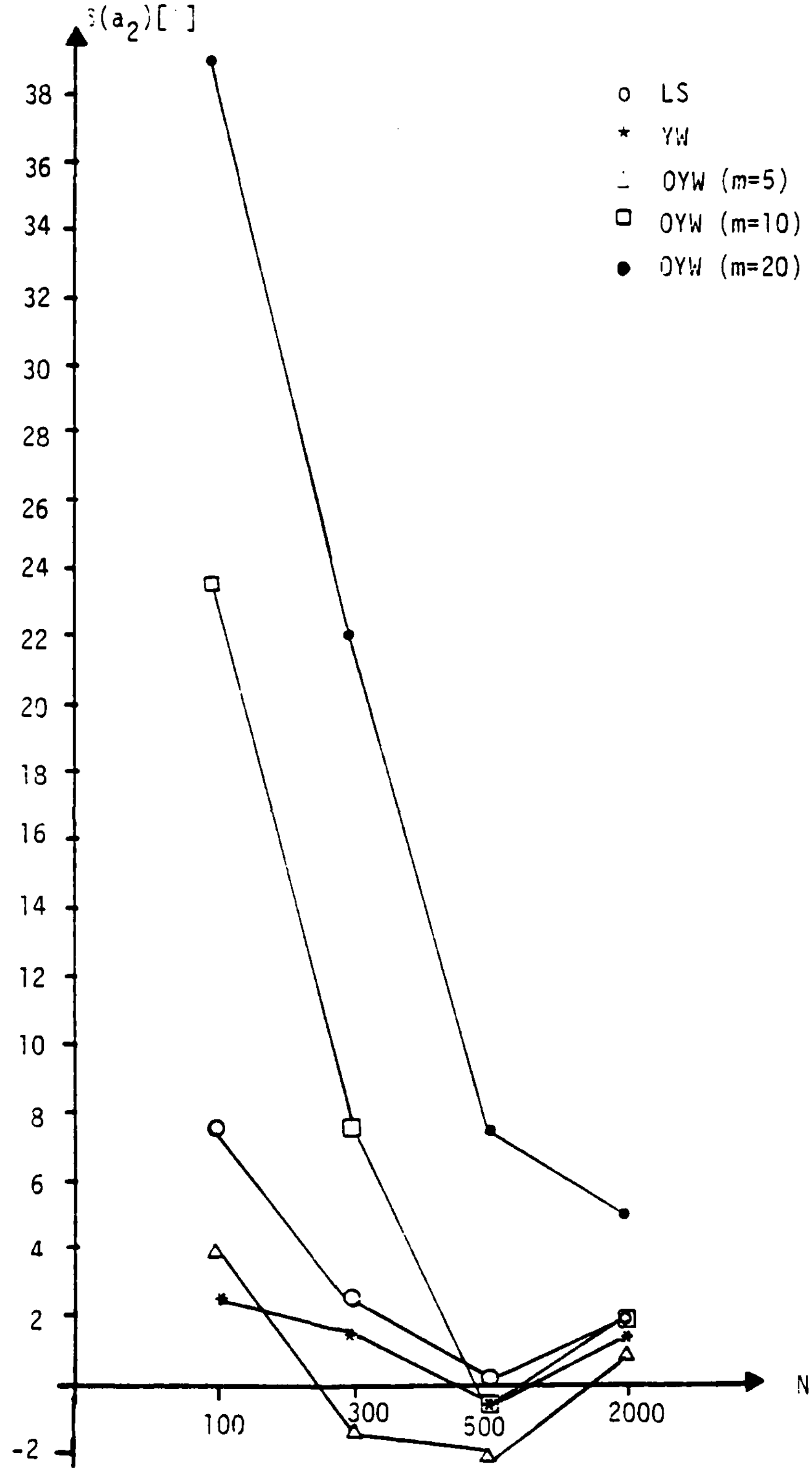

Figure 2B: Percentage Bias of $\hat{a}_2$ for S_1

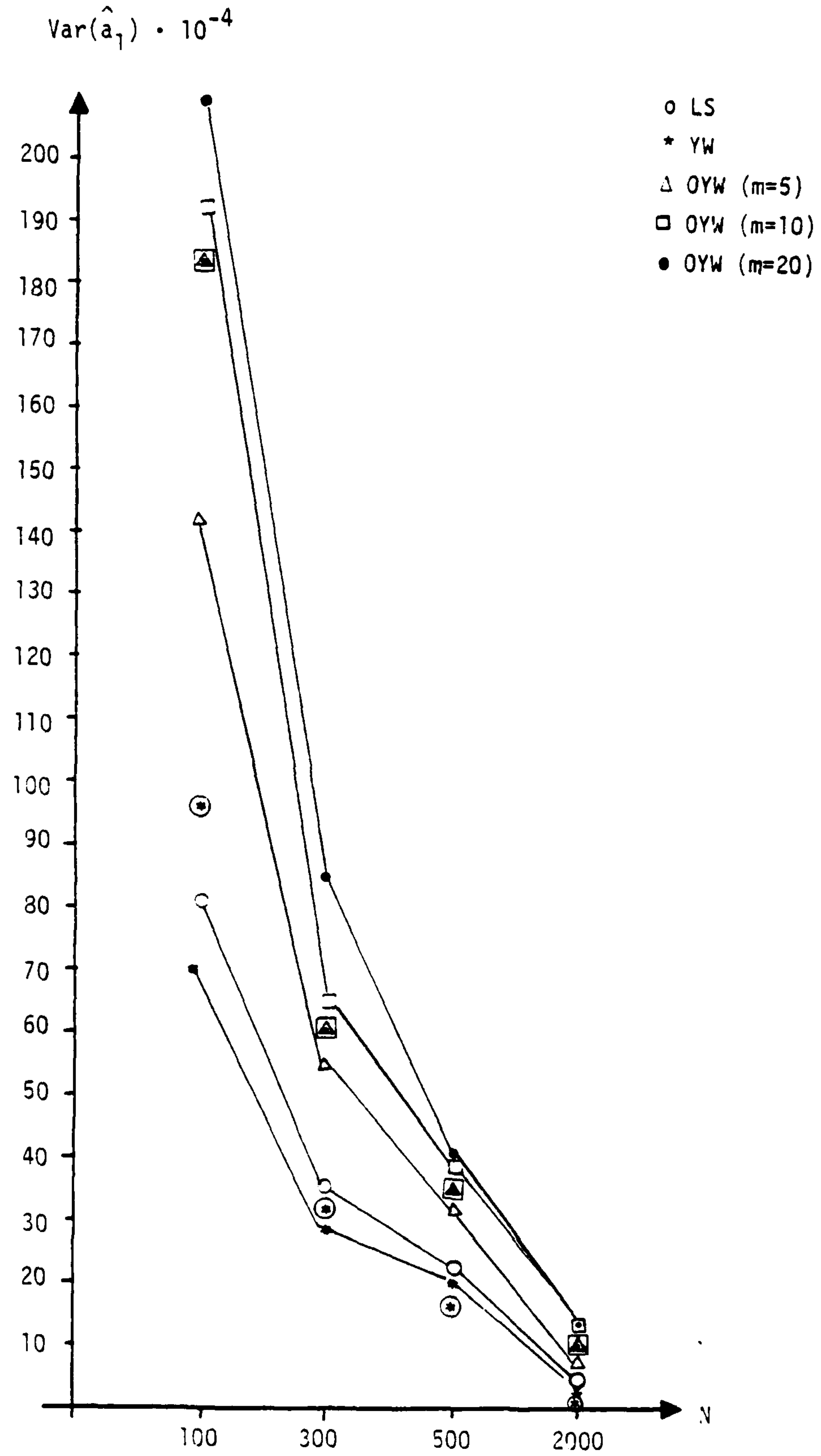

Figure 3A: Variance of $\hat{a}_1$ for S_1

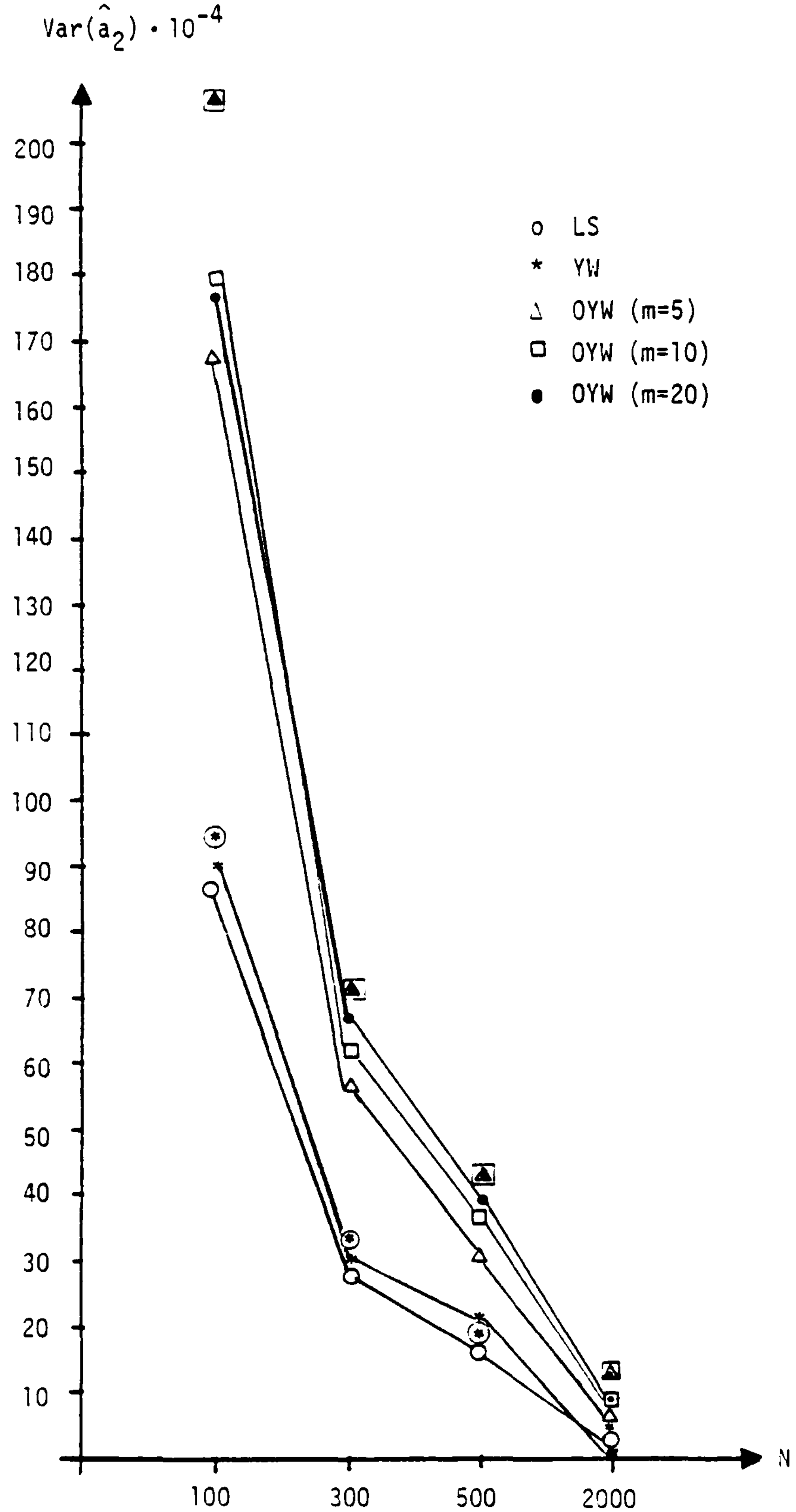

Figure 3B: Variance of $\hat{a}_2$ for S_1

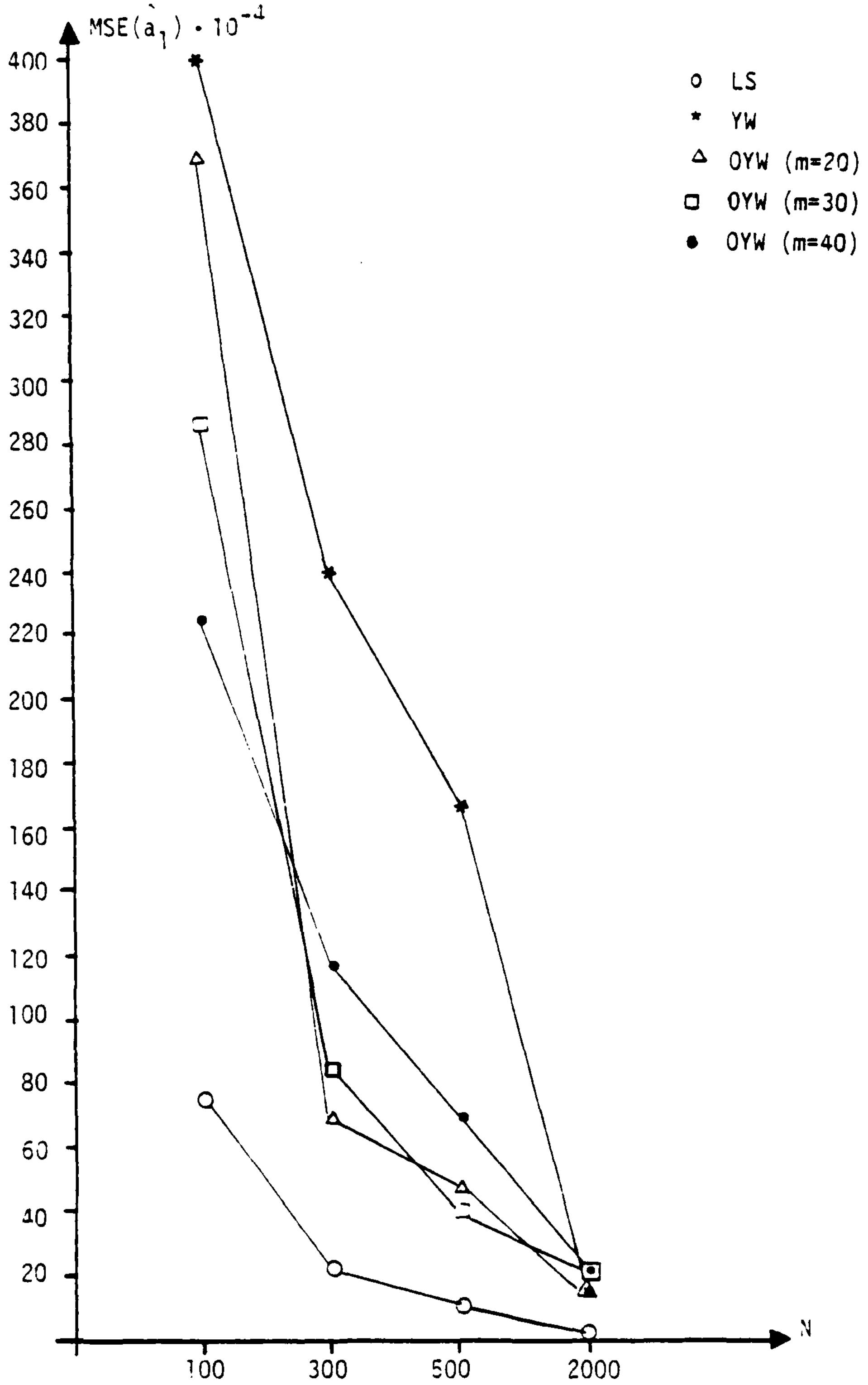

Figure 4A: Mean Square Error of $\hat{a}_1$ for S_2

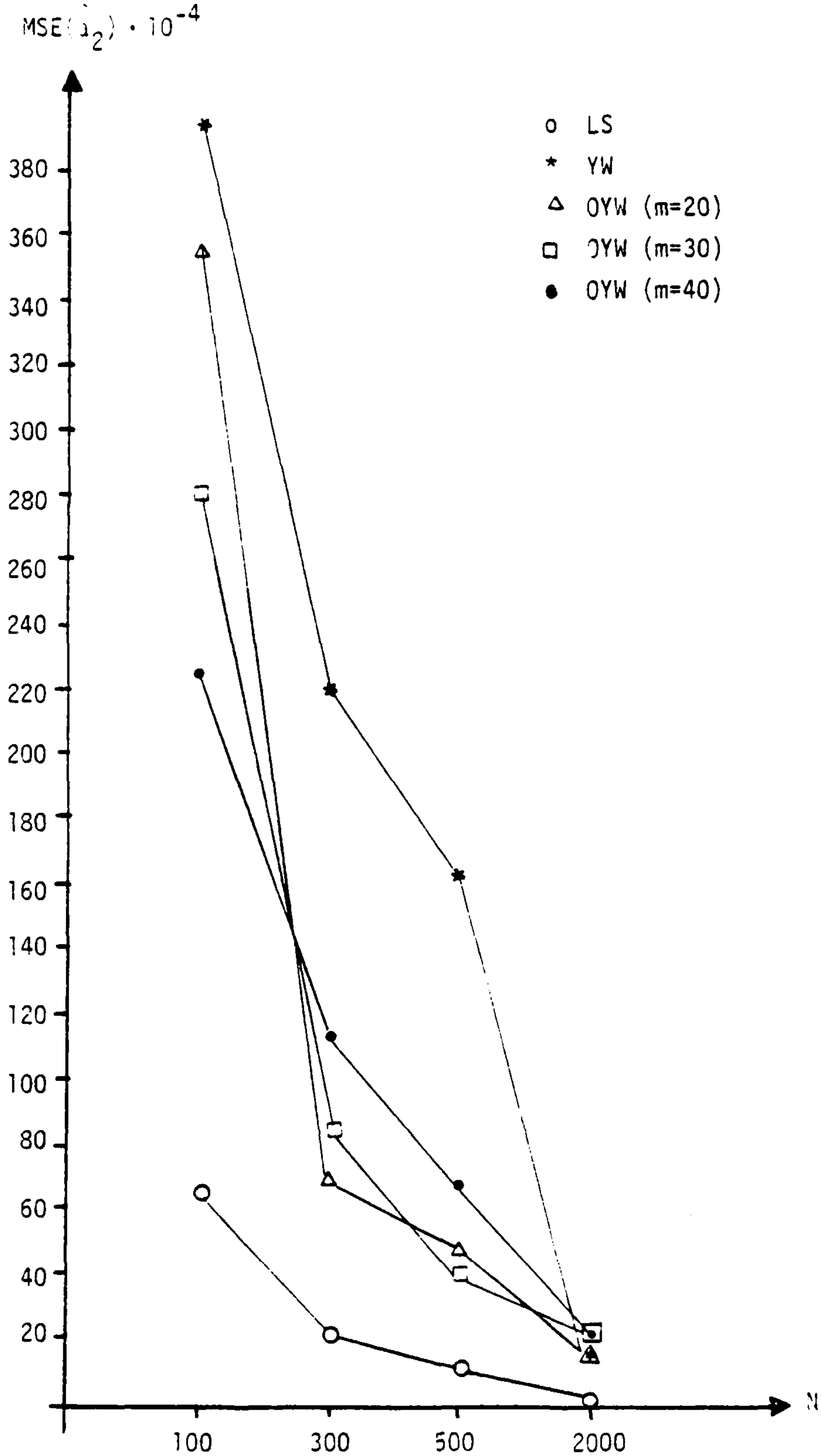

Figure 4B: Mean Square Error of $\hat{a}_2$ for S_2

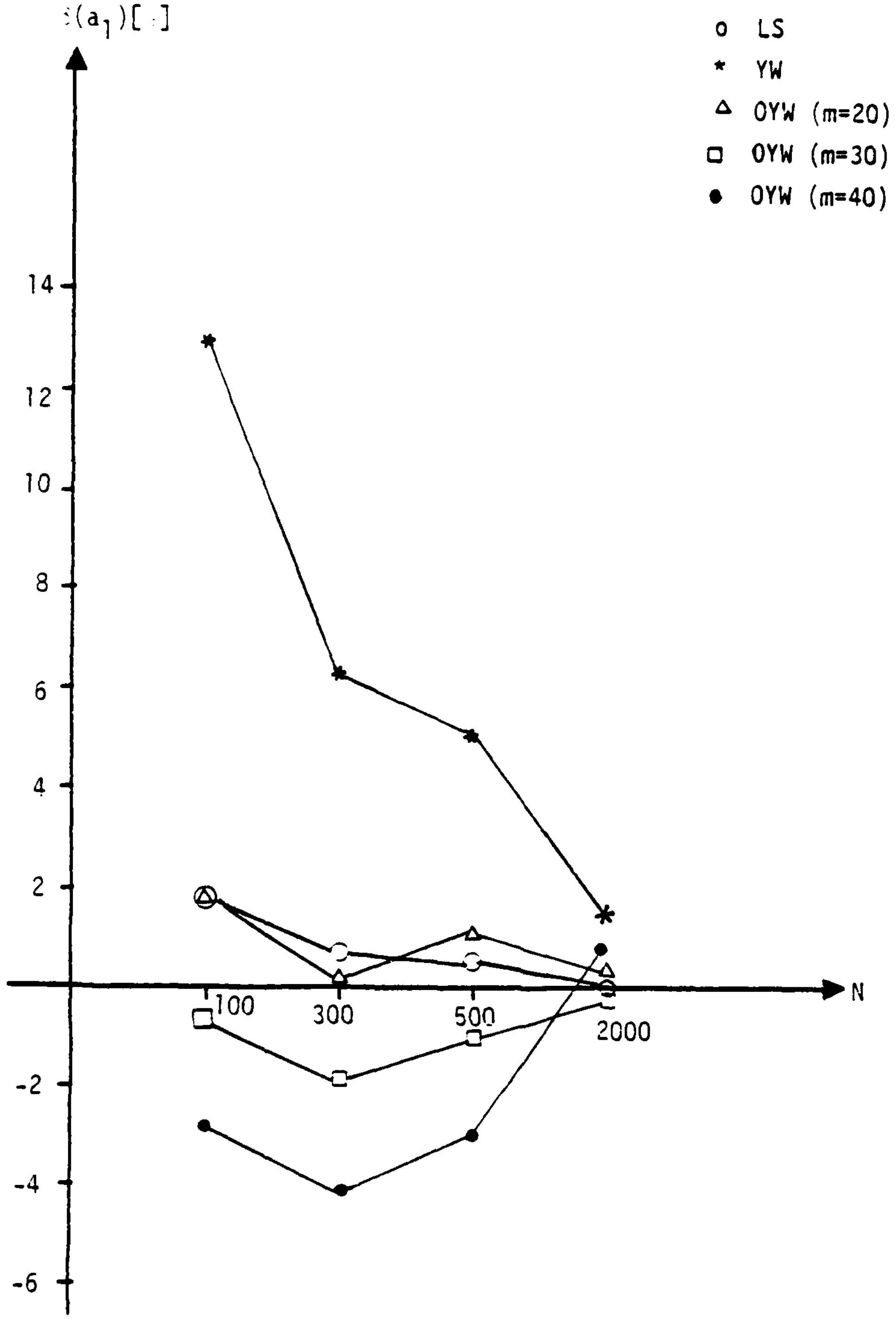

Figure 5A: Percentage Bias of $\hat{a}_1$ for S_2

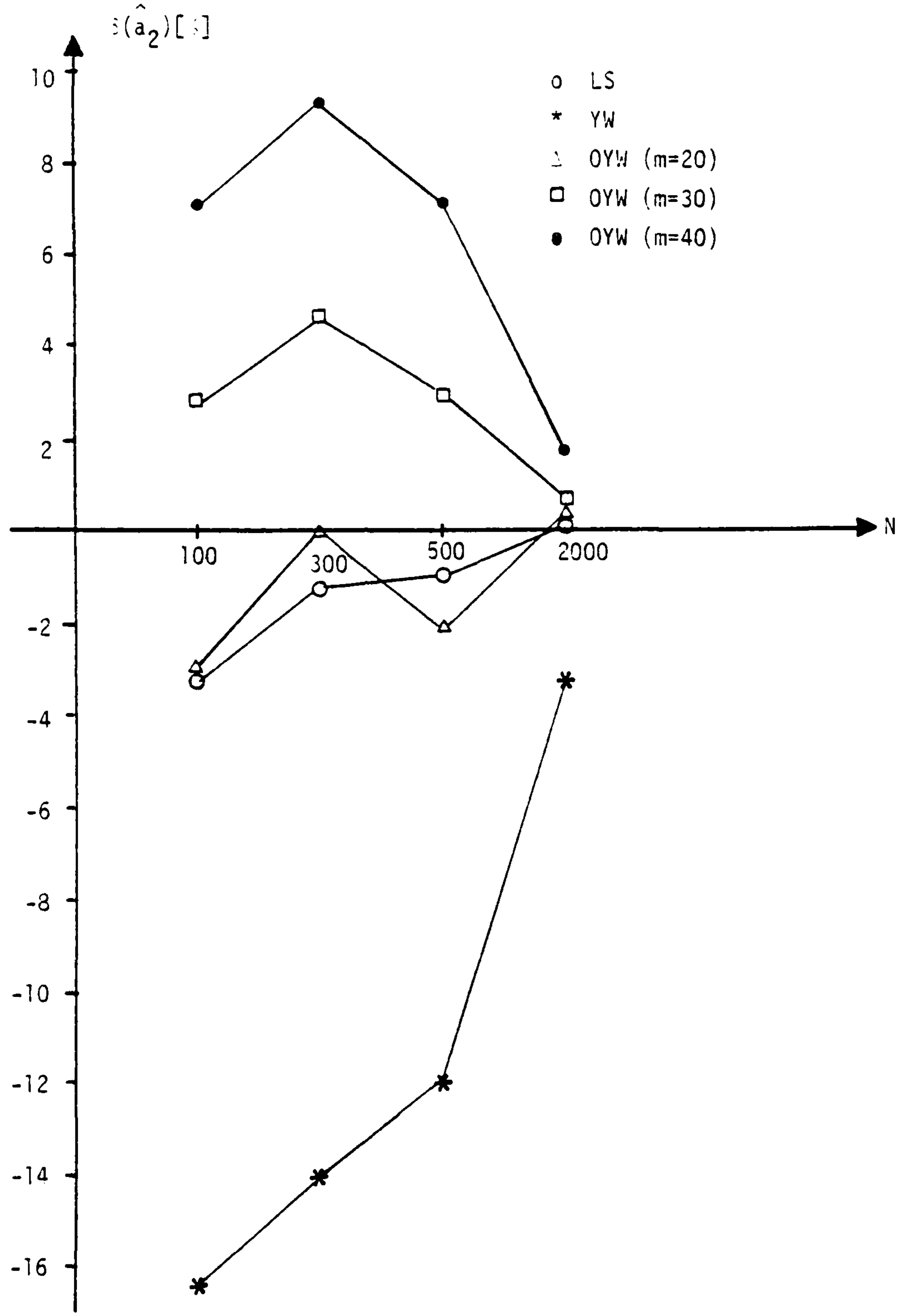

Figure 5B: Percentage Bias of $\hat{a}_2$ for S_2

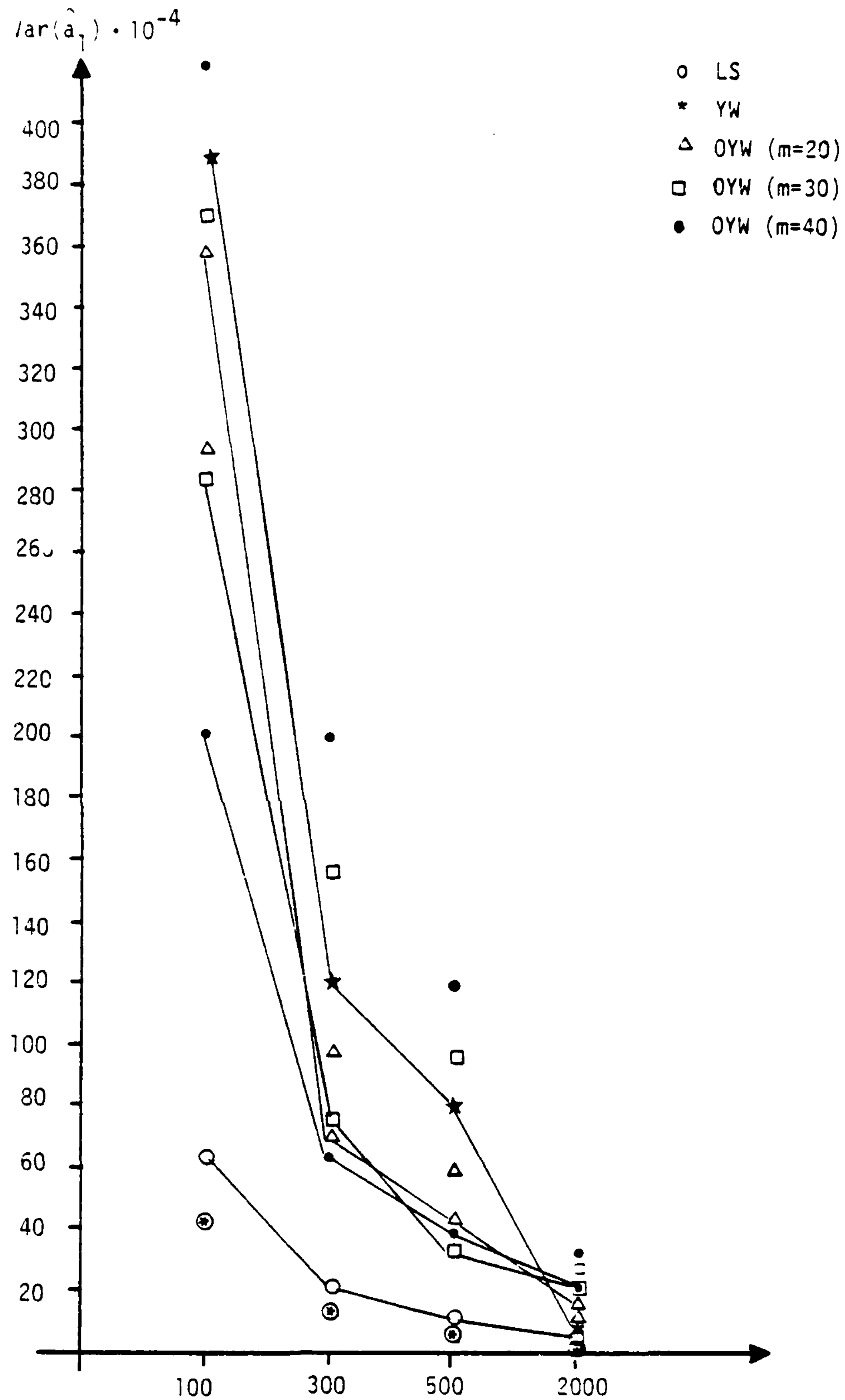

Figure 6A: Variance of $\hat{a}_1$ for S_2

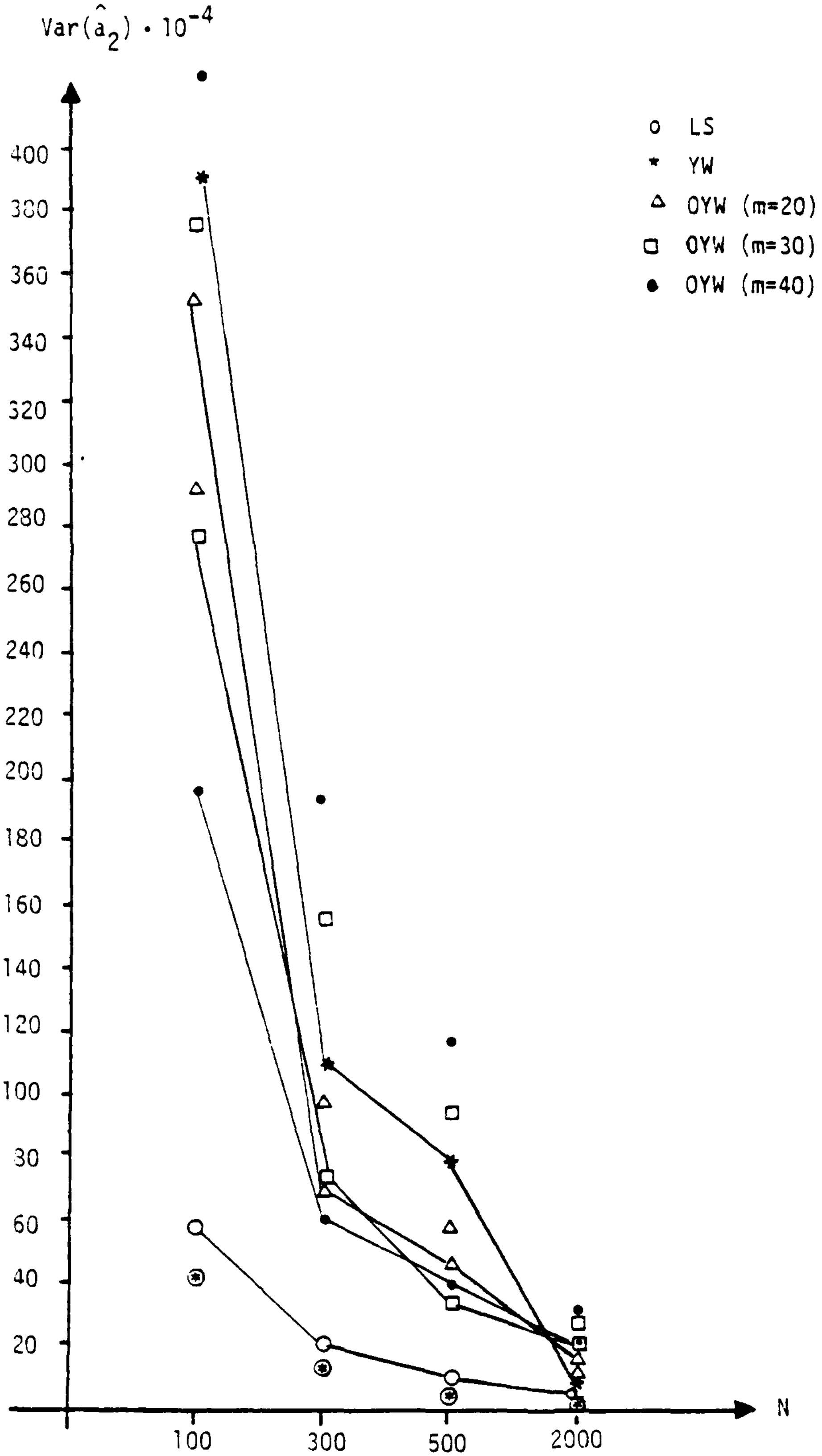

Figure 6B: Variance of $\hat{a}_2$ for S_2

APPENDIX D

COMPUTATION OF THE EXACT INFORMATION MATRIX OF GAUSSIAN TIME SERIES WITH STATIONARY RANDOM COMPONENTS

COMPUTATION OF THE EXACT INFORMATION MATRIX OF GAUSSIAN TIME SERIES WITH STATIONARY RANDOM COMPONENTS

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ABSTRACT

The paper presents an algorithm for efficient recursive computation of the Fisher information matrix of Gaussian time series whose random components are stationary, and whose means and covariances are functions of a parameter vector. The algorithm is first developed in a general framework and then specialized to the case of autoregressive moving-average processes, with possible additive white noise. The asymptotic behavior of the algorithm is explored and a termination criterion is derived. Finally, the algorithm is used to demonstrate the behavior the exact Cramer-Rao bound for some ARMA processes, as a function of the number of data points. It is shown that for processes with zeroes near the unit circle and short data records, the exact Cramer-Rao bound differs dramatically from its common approximation based on asymptotic theory.

This work was supported by the Army Research Office under Contract No. DAAG29-83-C-0027.

1. INTRODUCTION

A general time series $\{y_t\}$ can be decomposed as

$$y_t = x_t + m_t, \tag{1}$$

where $\{m_t\}$ is a deterministic sequence and $\{x_t\}$ is a zero-mean random sequence. In this paper we consider time series whose random components $\{x_t\}$ are stationary Gaussian processes. The joint probability density of N consecutive data points, say $\{y_0, y_1, \ldots, y_{N-1}\}$, is given by

$$f(\underline{y}) = (2\pi)^{-N/2}[\det R]^{-1/2}\exp\{-\tfrac{1}{2}[\underline{y}-\underline{m}]^T R^{-1}[\underline{y}-\underline{m}]\}, \tag{2}$$

where

$$\underline{y} = [y_0, y_1, \ldots, y_{N-1}]^T \qquad ; \ \underline{m} = [m_0, m_1, \ldots, m_{N-1}]^T,$$

and R is a Toeplitz matrix whose elements are the covariances of $\{x_t\}$, i.e.

$$(R)_{i,j} = r_{i-j} = E\{y_{t+i-j}\, y_t\} \ ; \quad 0 < i, j < N-1. \tag{3}$$

We now specialize our discussion to the case where the sequences $\{m_0, m_1, \ldots\}$ and $\{r_0, r_1, \ldots\}$ are functions of an M-dimensional vector θ . Such time series are said to be parametric, and θ is called the parameter vector.

Parametries Gaussian time series are very common in many statistical and engineering applications. As examples we mention autoregressive (AR) and autoregressive moving-average (ARMA) processes [1]. A problem of considerable

interest in parametric time series analysis is that of estimating the parameter vector θ from a set of N consecutive measurements. As is well known, the variance of any unbiased estimate $\hat{\theta}$ is bounded from below by the inverse of the Fisher information matrix, i.e.,

$$E\{\hat{\theta}\} = \theta \longrightarrow \mathrm{Var}\{\hat{\theta}\} > J^{-1}(\theta) \ , \tag{4}$$

where

$$[J(\theta)]_{k,\ell} = E\{\frac{\partial \log f(\underline{y})}{\partial \theta_k} \frac{\partial \log f(\underline{y})}{\partial \theta_\ell}\} \quad , \quad 1 < k, \ell < M \ . \tag{5}$$

For Gaussian time series, the Fisher information matrix is given by the expression

$$[J(\theta)]_{k,\ell} = \frac{1}{2} \mathrm{tr}\{R^{-1}(\theta) \frac{\partial R(\theta)}{\partial \theta_k} R^{-1}(\theta) \frac{\partial R(\theta)}{\partial \theta_\ell}\} + [\frac{\partial \underline{m}(\theta)}{\partial \theta_k}]^T R^{-1}(\theta) [\frac{\partial \underline{m}(\theta)}{\partial \theta_\ell}] \ , \tag{6}$$

where $\mathrm{tr}\{\cdot\}$ denotes the trace operator. While formula (6) is known, its proof does not appear to be readily available in the literature. We, therefore, provide a proof of this formula in Appendix A.

When the mean vector $\underline{m}$ is zero (or is independent of θ), and when the number of data points is sufficiently large, the information matrix can be approximated by Whittle's asymptotic formula [2]

$$[J(\theta)]_{k,\ell} = \frac{N}{4\pi} \int_{-\pi}^{\pi} \frac{\frac{\partial \phi(\omega)}{\partial \theta_k} \cdot \frac{\partial \phi(\omega)}{\partial \theta_\ell}}{\phi^2(\omega)} d\omega \ , \tag{7}$$

where $\phi(\omega)$ is the power spectral density function,

$$\phi(\omega) = r_0 + 2 \sum_{k=1}^{\infty} r_k \cos k\omega \; . \tag{8}$$

The use of Whittle's asymptotic formula is quite common in time series analysis. In particular, for ARMA processes this formula yields a relatively simple closed-form expression - see e.g. [1, p. 240-242]. However, the quality of this approximation (7) depends heavily on the nature of the process and on the number of data points, and may yield highly erroneous results if N is not sufficiently large.

Direct computation of (6) (assuming that the sequences $\{m_0, m_1, \ldots\}$, $\{r_0, r_1, \ldots\}$ and their partial derivatives are known) requires a number of operations proportional to N^3. In some cases it is desired to compute the values of $J(\theta)$ for all $1 < n < N$, in which case the total number of operations is proportional to N^4. This is probably one of the reasons why the exact formula (6) is not widely applied.

In this paper we derive an algorithm for recursive computation of the Fisher information matrix. The algorithm computes the information matrices for all $1 < n < N$ in a number of operations proportional to N^2. Thus, the algorithm is considerably more efficient than the direct use of formula (6). The algorithm is based on the well-known Levinson-Durbin algorithm for computing the orthogonal polynomials of a Toeplitz matrix.

The general algorithm is derived in section 2 of the paper. In section 3 we specialize it to some common rational parametric models. In section 4 we discuss the asymptotic behavior of the algorithm and give termination criteria. In section 5 we illustrate the use of the algorithm by some examples. It is shown that the exact CRB differs dramatically from the

asymptotic CRB in some cases.

The relative computational efficiency of the algorithm described here makes it possible to use the exact CRB for performance evaluation of ARMA estimation algorithms. The exact CRB provides a very useful reference point for studying and comparing various estimation procedures proposed in the literature. The fact that in some practical examples the exact CRB differs considerably from the asymptotic CRB motivates the use of the algorithm proposed here, rather than using the somewhat simpler asymptotic formulas.

2. THE ALGORITHM

Let the partial derivatives of $\{r_n\}$ and $\{m_n\}$ be denoted by

$$s_{n,k} = \frac{\partial r_n(\theta)}{\partial \theta_k} \quad ; \quad u_{n,k} = \frac{\partial m_n}{\partial \theta_k} \quad ; \quad 1 \le k \le M \tag{9}$$

The values of $\{r_n, m_n, s_{n,k}, u_{n,k} \; ; \; 0 \le n \le N-1, \; 1 \le k \le M\}$ are assumed to be available to the algorithm.

Let $\underline{\rho}_n$, $\underline{\sigma}_{n,k}$ and $\underline{\nu}_{n,k}$ denote the vectors

$$\underline{\rho}_n = [r_1, r_2, \ldots, r_n]^T \quad ; \quad \underline{\sigma}_{n,k} = [s_{1,k}, s_{2,k}, \ldots, s_{n,k}]^T$$

$$\underline{\nu}_{n,k} = [u_{0,k}, u_{1,k}, \ldots, u_{n,k}]^T .$$

Let R_n and $S_{n,k}$ be the $(n+1) \times (n+1)$ *Toeplitz matrices*

$$(R_n)_{i,j} = r_{i-j} \quad ; \quad (S_{n,k})_{i,j} = s_{i-j,k} \quad ; \quad 0 \le i,j \le n . \tag{10}$$

Let $\underline{\alpha}_n$ be the (n+1)-dimensional vector

$$\underline{\alpha}_n = \begin{array}{c} \begin{bmatrix} 1 \\ -R_{n-1}^{-1} \, \underline{\rho}_n \end{bmatrix} \end{array} \begin{array}{l} \} \; 1 \\ \} \; n \end{array} . \tag{11}$$

The components of $\underline{\alpha}_n$ are the coefficients of the n-th orthogonal (so-called Levinson-Szegö polynomial of the sequence $\{r_0, r_1, \ldots\}$ [3],[4].

Let $\tilde{I}$ denote the permutation matrix

$$\tilde{I} = \begin{bmatrix} 0 & & & 1 \\ & & 1 & \\ & \cdot^{\cdot^{\cdot}} & & 0 \\ 1 & & & \end{bmatrix} . \tag{12}$$

The dimension of $\tilde{I}$ will be always clear from the context. Also, for any vector $\underline{v}$ we denote

$$\tilde{\underline{v}} = \tilde{I}\, \underline{v} , \tag{13}$$

i.e., $\tilde{\underline{v}}$ is obtained from $\underline{v}$ by reversing the order of the components of the vector. Note the following property of the matrices R_n and $S_{n,k}$:

$$\tilde{I}\, R_n\, \tilde{I} = R_n \qquad , \tilde{I}\, S_{n,k}\, \tilde{I} = S_{n,k} . \tag{14}$$

Let us partition the matrix R_n in two ways, as follows

$$R_n = \begin{array}{c} \begin{bmatrix} r_0 & \underline{\rho}_n^T \\ \underline{\rho}_n & R_{n-1} \end{bmatrix} \begin{array}{l} \}\ 1 \\ \}\ n \end{array} \\ \begin{array}{cc} 1 \quad & \quad n \end{array} \end{array} = \begin{array}{c} \begin{bmatrix} R_n & \tilde{\underline{\rho}}_n \\ \tilde{\underline{\rho}}_n^T & r_0 \end{bmatrix} \begin{array}{l} \}\ n \\ \}\ 1 \end{array} \\ \begin{array}{cc} n \quad & \quad 1 \end{array} \end{array} . \tag{15}$$

Using the well known partitioned matrix inversion formula [5, pp.], we get

$$R_n^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & R_{n-1}^{-1} \end{bmatrix} + \delta_n^{-1} \underline{\alpha}_n \underline{\alpha}_n^T = \begin{bmatrix} R_{n-1}^{-1} & 0 \\ 0 & 0 \end{bmatrix} + \delta_n^{-1} \tilde{\underline{\alpha}}_n \tilde{\underline{\alpha}}_n^T , \tag{16}$$

where

$$\delta_n = r_0 - \underline{\rho}_n^T R_{n-1}^{-1} \underline{\rho}_n = r_0 - \bar{\underline{\rho}}_n^T \bar{R}_{n-1}^{-1} \bar{\underline{\rho}}_n \quad . \tag{17}$$

Consider now the (k,ℓ)-th element of the Fisher information matrix $J_{n+1}(0)$ corresponding to the n+1 measurements $\{y_0, y_1, \ldots, y_n\}$. Using (6) and (16), we can expand this element as follows:

$$[J_{n+1}(\theta)]_{k,\ell} = \frac{1}{2} \operatorname{tr}\{R_n^{-1} S_{n,k} R_n^{-1} S_{n,\ell}\} + \underline{\nu}_{n,k}^T R_n^{-1} \underline{\nu}_{n,\ell}$$

$$= \frac{1}{2} \operatorname{tr}\left\{ \begin{bmatrix} 0 & 0 \\ 0 & R_{n-1}^{-1} \end{bmatrix} S_{n,k} \begin{bmatrix} 0 & 0 \\ 0 & R_{n-1}^{-1} \end{bmatrix} S_{n,\ell} \right\}$$

$$+ \frac{1}{2} \delta_n^{-1} \operatorname{tr}\left\{ \underline{\alpha}_n \underline{\alpha}_n^T S_{n,k} \begin{bmatrix} 0 & 0 \\ 0 & R_{n-1}^{-1} \end{bmatrix} S_{n,\ell} \right\}$$

$$+ \frac{1}{2} \delta_n^{-1} \operatorname{tr}\left\{ \begin{bmatrix} 0 & 0 \\ 0 & R_{n-1}^{-1} \end{bmatrix} S_{n,k} \underline{\alpha}_n \underline{\alpha}_n^T S_{n,\ell} \right\}$$

$$+ \frac{1}{2} \delta_n^{-2} \operatorname{tr}\{\underline{\alpha}_n \underline{\alpha}_n^T S_{n,k} \underline{\alpha}_n \underline{\alpha}_n^T S_{n,\ell}\}$$

$$+ \underline{\nu}^T_{n,k} \begin{bmatrix} R^{-1}_{n-1} & 0 \\ 0 & 0 \end{bmatrix} \underline{\nu}_{n,\ell} + \delta_n^{-1} \underline{\nu}_{n,k} \bar{\underline{\alpha}}_n \bar{\underline{\alpha}}_n^T \underline{\nu}_{n,\ell}$$

$$= [J_n(\theta)]_{k,\ell} + \delta_n^{-1} \underline{\alpha}_n^T S_{n,k} \begin{bmatrix} 0 & 0 \\ 0 & R^{-1}_{n-1} \end{bmatrix} S_{n,\ell} \underline{\alpha}_n$$

$$+ \frac{1}{2} \delta_n^{-2} \underline{\alpha}_n^T S_{n,k} \underline{\alpha}_n \underline{\alpha}_n^T S_{n,\ell} \underline{\alpha}_n + \delta_n^{-1} \underline{\nu}^T_{n,k} \bar{\underline{\alpha}}_n \bar{\underline{\alpha}}_n^T \underline{\nu}_{n,\ell}$$

$$= [J_n(\theta)]_{k,\ell} + \delta_n^{-1} \underline{\alpha}_n^T S_{n,k} R_n^{-1} S_{n,\ell} \underline{\alpha}_n - \frac{1}{2} \delta_n^{-2} \underline{\alpha}_n^T S_{n,k} \underline{\alpha}_n \underline{\alpha}_n^T S_{n,\ell} \underline{\alpha}_n$$

$$+ \delta_n^{-1} \underline{\nu}^T_{n,k} \bar{\underline{\alpha}}_n \bar{\underline{\alpha}}_n^T \underline{\nu}_{n,\ell} \; . \tag{18}$$

Let us introduce following auxiliary variables.

$$\underline{\eta}_{n,k} = S_{n,k} \underline{\alpha}_n \; , \tag{19a}$$

$$\underline{\xi}_{n,k} = R_n^{-1} S_{n,k} \underline{\alpha}_n = R_n^{-1} \underline{\eta}_{n,k} \; , \tag{19b}$$

$$f_{n,k} = \underline{\alpha}_n^T S_{n,k} \underline{\alpha}_n = \underline{\alpha}_n^T \underline{\eta}_{n,k} \; , \tag{19c}$$

$$g_{n,k} = \bar{\underline{\alpha}}_n \underline{\nu}_{n,k} \; . \tag{19d}$$

Substituting in (18) we get the desired update formula for $J_{n+1}(\theta)$:

$$[J_{n+1}(\theta)]_{k,\ell} = [J_n(\theta)]_{k,\ell} + \delta_n^{-1} \underline{\eta}_{n,k}^T \underline{\xi}_{n,\ell} - \frac{1}{2}\delta_n^{-2} f_{n,k} f_{n,\ell} + \delta_n^{-1} g_{n,k} g_{n,\ell} \; . \tag{20}$$

To use this formula in a recursive algorithm, we need updating recursions for $\underline{\alpha}_n$, δ_n, $\underline{\eta}_{n,k}$ and $\underline{\xi}_{n,k}$. The variables $\underline{\alpha}_n$ and δ_n can be update using the Levinson-Durbin recursions.

$$\underline{\alpha}_n = \begin{bmatrix} \underline{\alpha}_{n-1} \\ 0 \end{bmatrix} - c_n \begin{bmatrix} 0 \\ \tilde{\underline{\alpha}}_{n-1} \end{bmatrix} , \tag{21}$$

$$\delta_n = \delta_{n-1}(1 - c_n^2) \; , \tag{22}$$

where c_n is the n-th order partial correlation (also called reflection coefficient)

$$c_n = \delta_{n-1}^{-1} \tilde{\underline{\alpha}}_{n-1}^T \underline{\rho}_n \; . \tag{23}$$

Using (21) in (19a), we get the following update formula for $\underline{\eta}_{n,k}$:

$$\underline{\eta}_{n,k} = S_{n,k} \underline{\alpha}_n = S_{n,k} \begin{bmatrix} \underline{\alpha}_{n-1} \\ 0 \end{bmatrix} - c_n S_{n,k} \begin{bmatrix} 0 \\ \tilde{\underline{\alpha}}_{n-1} \end{bmatrix}$$

$$= \begin{bmatrix} \underline{\eta}_{n-1,k} \\ d_{n,k} \end{bmatrix} - c_n \begin{bmatrix} d_{n,k} \\ \bar{\eta}_{n-1,k} \end{bmatrix} , \tag{24}$$

where

$$d_{n,k} = \underline{\bar{\alpha}}_{n-1}^{T} \underline{\sigma}_{n,k} . \tag{25}$$

Similarly we get for $\underline{\xi}_{n,k}$,

$$\underline{\xi}_{n,k} = R_n^{-1} \underline{\eta}_{n,k} = R_n^{-1} \{ \begin{bmatrix} \underline{\eta}_{n-1,k} \\ d_{n,k} \end{bmatrix} - c_n \begin{matrix} d_{n,k} \\ \bar{\underline{\eta}}_{n-1,k} \end{matrix} \}$$

$$= \{ \begin{bmatrix} R_{n-1}^{-1} & 0 \\ 0 & 0 \end{bmatrix} + \delta_n^{-1} \bar{\underline{\alpha}}_n \underline{\bar{\alpha}}_n^{T} \} \begin{bmatrix} \underline{\eta}_{n-1,k} \\ d_{n,k} \end{bmatrix}$$

$$-c_n \{ \begin{bmatrix} 0 & 0 \\ 0 & R_{n-1}^{-1} \end{bmatrix} + \delta_n^{-1} \underline{\alpha}_n \, \alpha_n^{T} \} \begin{bmatrix} d_{n,k} \\ \bar{\underline{\eta}}_{n-1,k} \end{bmatrix}$$

$$= \begin{bmatrix} \bar{\underline{\xi}}_{n-1,k} \\ 0 \end{bmatrix} - c_n \begin{bmatrix} 0 \\ \bar{\underline{\xi}}_{n-1,k} \end{bmatrix} + e_{n,k} \delta_n^{-1} \{ \bar{\underline{\alpha}}_n - c_n \underline{\alpha}_n \} , \tag{26}$$

where

$$e_{n,k} = \underline{\alpha}_n^{-T} \begin{bmatrix} \eta_{n-1,k} \\ d_{n,k} \end{bmatrix} . \qquad (27)$$

To summarize, the algorithm consists of formulas (23), (25), (22), (21), (27), (24), (26), (19c), (19d) and (18) in the given order. For convenience, we have included a summary of the algorithm in appendix B, in a form readily adaptable for programming. The total operation count for one update is $n(M^2 + 6M + 4) + (M^2 + 2M + 1)$ multiply/divide operations, and $n(M^2 + 6M + 4) + (3M^2 + 6M + 3)$ add/subtract operations. Thus, the total operation count for computing $\{J_n(\theta),\ 1 < n < N\}$ is $\frac{1}{2}(M^2 + 6M + 4)N^2 + \frac{1}{2}(3M^2 + 10M + 6)N$ multiply/divide oeprations, and $\frac{1}{2}(M^2 + 6M + 4)N^2 + \frac{1}{2}(7M^2 + 18M + 10)N$ add/subtract operations. This does not include the computation of $\{r_n,\ s_{n,k},\ m_n,\ u_{n,k}\}$, which depends on the specific parametric model.

3. RATIONAL PARAMETRIC TIME SERIES

In this section we present an algorithm for computing the covariances of rational parametric models, and their partial derivatives. We consider the following general rational model:

$$y_t = z_t + v_t , \tag{28}$$

where $\{v_t\}$ is white Gaussian noise with zero mean and variance σ_v^2, and z_t is a (p,q) ARMA process,

$$z_t = -\sum_{k=1}^{p} a_k z_{t-k} + u_t + \sum_{k=1}^{q} b_k u_{t-k} , \tag{29}$$

where $\{u_t\}$ is white Gaussian noise with zero mean and variance σ_u^2. The random processes $\{u_t\}$ and $\{v_t\}$ are assumed to be uncorrelated. The parameter vector is

$$\theta = [\sigma_u^2, a_1, \ldots, a_p, b_1, \ldots, b_q, \sigma_v^2]^T \quad ; \quad M = p + q + 2 . \tag{30}$$

The model defined by (28), (29) includes many common rational models as special cases. The case $\sigma_v^2 = 0$, $q = 0$ corresponds to a pure AR process, while the case $\sigma_v^2 \neq 0$, $q = 0$ corresponds to an AR processes in additive white noise. The case $p = 0$, $\sigma_v^2 = 0$ corresponds to a pure MA process, while the case $p \neq 0$, $q \neq 0$ corresponds to an ARMA process. Note that in general, the additive noise v_t is redundant whenever $q > p$, because then it can be absorbed in z_t by a proper modification of the parameter $\{b_k\}$.

The covariances of $\{y_t\}$ can be computed as follows. Let us introduce the auxiliary AR process

$$w_t = -\sum_{k=1}^{p} a_k w_{t-k} + u_t/\sigma_u \ . \tag{31}$$

Let $\{\gamma_n\}$ denote the covariance sequence of the process $\{w_t\}$. The first p+1 elements of this sequence can be obtained by solving the equations [5]

$$(A_1+A_2)[\gamma_p, \ \ldots, \ \gamma_1, \tfrac{1}{2}\gamma_0]^T = [0 \ \ldots \ 0 \quad 1]^T, \tag{32}$$

where

$$A_1 = \begin{bmatrix} 1 & a_1 & \cdots & a_p \\ & \ddots & \ddots & \vdots \\ & & \ddots & a_1 \\ 0 & & & 1 \end{bmatrix} ; \quad A_2 = \begin{bmatrix} 0 & & & a_p \\ & & \iddots & \vdots \\ & \iddots & & a_1 \\ a_p & \cdots & a_1 & 1 \end{bmatrix} . \tag{33}$$

The higher order elements of the sequence $\{\gamma_n\}$ can be computed using the recursion

$$\gamma_n = -\sum_{k=1}^{p} a_k \gamma_{n-k} \quad ; \quad n > p \ . \tag{34}$$

The covariances of $\{y_t\}$ are related to those of $\{w_t\}$ via

$$r_n = \sigma_u^2 \cdot \sum_{i=0}^{q} \sum_{j=0}^{q} b_i b_j \gamma_{|n-i+j|} + \begin{cases} \sigma_v^2 & ; \ n=0 \\ 0 & ; \ \text{otherwise} \end{cases} . \tag{35}$$

The partial derivatives of the covariances can be similarly computed.

Differentiating (32) with respect to a_k we get

$$(A_1+A_2)\left[\frac{\partial\gamma_p}{\partial a_k},\ldots,\frac{\partial\gamma_1}{\partial a_k},\frac{1}{2}\frac{\partial\gamma_0}{\partial a_k}\right]^T = [\gamma_{p-k},\ldots,\gamma_0,\ldots,\gamma_k]^T . \tag{36}$$

By solving these equations for each $1 \leq k \leq p$, we get $\{\frac{\partial\gamma_0}{\partial a_k},\ldots,\frac{\partial\gamma_p}{\partial a_k}, 1 \leq k \leq p\}$ (note that A_1+A_2 needs to be inverted only once). Differentiating (34) with respect to a_k we get

$$\frac{\partial\gamma_n}{\partial a_k} = -\sum_{i=1}^{p} a_i \frac{\partial\gamma_{n-i}}{\partial a_k} - \gamma_{n-k} . \tag{37}$$

Finally,

$$\frac{\partial r_n}{\partial a_k} = \sigma_u^2 \sum_{i=0}^{q}\sum_{j=0}^{q} b_i b_j \frac{\partial\gamma_{|n-i+j|}}{\partial a_k} \tag{38a}$$

$$\frac{\partial\gamma_n}{\partial b_k} = \sigma_u^2 \sum_{i=0}^{q} b_i(\gamma_{|n-k+i|} + \gamma_{|n+k-i|}) \tag{38b}$$

$$\frac{\partial\gamma_n}{\partial\sigma_u^2} = \sum_{i=0}^{q}\sum_{j=0}^{q} b_i b_j \gamma_{|n-i+j|} \tag{38c}$$

$$\frac{\partial\gamma_n}{\partial\sigma_v^2} = \begin{cases} 1 & ; \quad n=0 \\ 0 & ; \quad \text{otherwise} \end{cases} . \tag{38d}$$

Equations (32), (34), (35), (36), (37), (38) provide an algorithm for computing the covariances and their partial derivatives for the ARMA plus noise model.

4. ASYMPTOTIC BEHAVIOR OF THE ALGORITHM

In this section we limit ourselves to time series with zero meanss. Furthermore, we assume that $\{y_t\}$ has nonzero innovation variance σ^2. By Whittle's formula (7), the information matrix $J_n(\theta)$ is asymptotically proportional to n. We therefore expect that the increment $\delta_n^{-1}\underline{\eta}_{n,k}^T\underline{\xi}_{n,\ell} - \frac{1}{2}\delta_n^{-2} f_{n,k}\, f_{n,\ell}$ appearing in the update formula (20) will converge to a constant value as n goes to infinity. Indeed, recall that

$$\sigma^2 = \lim_{n\to\infty} \delta_n = r_0 \prod_{i=1}^{\infty} (1-c_i^2) \ . \tag{39}$$

Hence,

$$\sum_{i=1}^{\infty} c_i^2 < - \sum_{i=1}^{\infty} \log(1-c_i^2) = \log \frac{r_0}{\sigma^2} < \infty \ , \tag{40}$$

Therefore,

$$\lim_{n\to\infty} \sum_{i=n+1}^{\infty} c_i^2 = 0 \ . \tag{41}$$

It is easy to show that due to (40) the variables δ_n, $f_{n,k}$ and $\underline{\eta}_{n,k}^T\,\underline{\xi}_{n,e}$ converge to constant values as $n \to \infty$. Thus, for large enough n, $J_n(\theta)$ becomes approximately linear in n. This means that for some n_0,

$$J_n(\theta) \approx {}_{n_0}(\theta) + (n-n_0)\,\bar{J}(\theta) \qquad , \quad n > n_0 \ , \tag{42}$$

where $\bar{J}(\theta)$ is a constant matrix. The approximate relationship (42) can be used to terminate the information updating algorithm, in the following manner. Suppose we can find some n_0 such that

$$\sum_{i=n_0+1}^{\infty} c_i^2 < \varepsilon << 1 , \tag{43}$$

where ε is determined by the desired degree of accuracy. Then we can stop the algorithm at $n=n_0$, take $\mathcal{J}(\theta)$ as the last computed increment of $J_n(\theta)$, and extrapolate $J_n(\theta)$ for all $n > n_0$ using (42). The problem is, of course, to determine n_0 so as to guarantee (43). One way of doing this is to compute a moving sum of squared partial correlations, say

$$K_n = \sum_{i=n-n_1}^{n} c_i^2 , \tag{44}$$

where n_1, is fixed. Then we can choose n_0 as the first value of n for which $K_n < \varepsilon$. If the sequence $\{c_n\}$ is sufficiently regular, this criterion is a reasonable approximation of (43).

For ARMA processes, the value of n_0 can be determined by the ARMA parameters, and there is no need to actually test the partial conrrelations. In fact, a consecutive estimate of n_0 is provided by the following lemma.

Lemma:

Let $\sigma b(z)/a(z)$ be the transfer function of a (p,q) ARMA process, and assume that all the roots of b(z), $\{\beta_k, 1 < k < q\}$, are inside the unit circle. Let n_1 be an integer such that

$$\sum_{k=1}^{q} |\beta_k|^{n_1} < \frac{1}{2} \varepsilon , \tag{45}$$

and let

$$n_0 = p + q(n_1-1) . \tag{46}$$

Then

$$\sum_{i=n_0+1}^{\infty} c_i^2 < \varepsilon .$$

Proof:

Let $\alpha_n(z)$ be the n-th order Levinson-Szegö polynomial of the given process (i.e., the polynomial whose coefficients are the components of $\underline{\alpha}_n$) . As is well known, $\alpha_n(z)$ minimizes the prediction error variance among all polynomials of degree n, and the minimum prediction error variance is δ_n . Therefore,

$$\delta_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \frac{\alpha_n(e^{j\omega})\sigma b(e^{j\omega})}{a(e^{j\omega})} \right|^2 d\omega \le \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \frac{\lambda(e^{j\omega})\sigma b(e^{j\omega})}{a(e^{j\omega})} \right|^2 d\omega , \tag{47}$$

for any n-th degree polynomial $\lambda(z)$. Let us deinfe the following polynomial:

$$\lambda_0(z) = a(z) \cdot \prod_{k=1}^{q} (1+\beta_k z^{-1} + \ldots + \beta_k^{n_1-1} z^{-(n_1-1)}). \tag{48}$$

By (46), the degree of $\lambda_0(z)$ is n_0 . Hence by (47),

$$\delta_{n_0} \le \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \frac{\sigma\lambda_0(e^{j\omega})b(e^{j\omega})}{a(e^{j\omega})} \right|^2 d\omega = \frac{\sigma^2}{2\pi} \int_{-\pi}^{\pi} \left| \prod_{k=1}^{q} (1-(\beta_k e^{-j\omega})^{n_1} \right| d\omega$$

$$\le \sup_{-\pi \le \omega \le \pi} \sigma^2 \left| \prod_{k=1}^{q} (1-(\beta_k e^{-j\omega})^{n_1}) \right|^2 = \sigma^2 [\prod_{k=1}^{q} (1+|\beta_k|^{n_1})]^2 . \tag{49}$$

Hence, by (45),

$$\log \frac{\delta_{n_0}}{\sigma^2} < \sum_{k=1}^{q} 2\log(1+|\beta_k|^{n_1}) < 2\sum_{k=1}^{q} |\beta_k|^{n_1} < \varepsilon . \tag{50}$$

Finally,

$$\sum_{i=n_0+1}^{\infty} c_i^2 < \sum_{i=n_0+1}^{\infty} - \log(1-c_i^2) = \log \frac{\delta_{n_0}}{\sigma^2} < \varepsilon . \tag{51}$$

In summary, any n_0 that satisfies (45), (46) can be used as a termination point for the algorithm. Note that n_0 is essentially determined by the zero(es) using the largest magnitude. The denominator polynomial has little effect on n_0 , except when $p >> q$ or when all the zeroes of b(z) have small magnitudes. For pure AR processes, $c_i = 0$ for all $i > p$. The relationship (42) then holds exactly for $n_0 = p+1$. This fact was also proven in [6] using different arguments. For ARMA processes having zeroes near the unit circle, partial correlations may converge to zero very slowly. Therefore, for such processes, the Fisher information matrix reaches its asymptotic approximation (7) only at very large values of n.

5. NUMERICAL EXAMPLES

In this section we illustrate the behavior of the Fisher information matrix of ARMA processes by some examples. Rather than considering the information matrix itself, we consider the following quantities:

(i) The diagonal elements of $J_n^{-1}(\theta)$; these are the Cramer-Rao bounds on the respective components of the parameter vector θ .

(ii) The Cramer-Rao bound for unbiased estimates of the logarithm of the spectral density. this is given by [7]

$$\mathrm{CRB}\{\log\phi(\omega)\} = D^T(\omega)J_n^{-1}(\theta)D(\omega), \tag{52}$$

where

$$D^T(\omega) = \frac{1}{\phi(\omega)}\left[\frac{\partial\phi(\omega)}{\partial\theta_1}, \ \ldots, \ \frac{\partial\phi(\omega)}{\partial\theta_n}\right] . \tag{53}$$

<u>Example #1:</u>

In this example we consider an ARMA processof order (2,2), with a pair of conjugate poles and a pair of conjugate zeroes. Both the poles and the zeroes have magnitudes $(0.95)^{1/2}$, and the phase angles are ±45° for the poles and ±135° for the zeroes. The ARMA transfer function is

$$\frac{b(z)}{a(z)} = \frac{1+1.378z^{-1}+0.95z^{-2}}{1-1.378z^{-1}+0.95z^{-2}} . \tag{54}$$

Figure 1a shows the CRB's of the parameters a_1, a_2, b_1, b_2 as a function of n. The CRB's are in dB and the n axis is in log scale, so that the asymptotic

approximations appear as straight lines. As can be seen, the discrepancy between the exact bounds and the asymptotic approximations is very large when the number of data points is small, especially for the numerator parameters. Only at about $n = 500$ do the exact bounds converge to their asymptotic approximations. Figures 1b and 1c show the exact and asymptotic bounds on the log spectrum (±1 standard deviation) for 50 data points. As can be seen, the behavior of the bound in the vicinity of the pole is similar in both figures. However, its behavior in the vicinity of the zero is considerably different: the asymptotic approximation is far too optimistic.

Example #2:

This example is similar to the previous one, except that the poles were moved to phase angles of ±70° and the zeroes were moved to phase angles of ±110°. The corresponding transfer function is

$$\frac{b(z)}{a(z)} = \frac{1+0.667z^{-1}+0.95z^{-2}}{1-0.667z^{-1}+0.95z^{-2}} \quad . \tag{55}$$

Figure 2a shows the bounds on the parameters, Figure 2b shows the exact bounds on the spectrum, and Figure 2c depicts the approximate bounds on the spectrum. Note the difference in the bound of compared to the previous example.

Example #3:

Here we moved the pole and the zero even closer to each other. The poles have phase angles of ±85° and the zeroes have phase angles of ±95°. The corresponding transfer function is

$$\frac{b(z)}{a(z)} = \frac{1+0.17z^{-1}+0.95z^{-2}}{1-0.17z^{-1}+0.95z^{-2}} . \tag{56}$$

The bounds are shown in Figures 3a, 3b, 3c. Note the dramatic change in the bounds of a_1 and a_2 compared to the previous examples, for small values of n.

Example #4:

For this and the subsequent examples the model was a sum of two uncorrelated narrowband processes and white noise. Such a process has a spectral density function

$$\phi(\omega) = \sum_{i=1}^{2} \frac{E_i[(1-\rho_i^4) - (1-\rho_i^2)\rho_i\cos 2\pi f_i(e^{j\omega}+e^{-j\omega})}{1+4\rho_i^2\cos^2 2\pi f_i + \rho_i^4 - 2\rho_i\cos 2\pi f_i(1+\rho_i^2)(e^{j\omega}+e^{-j\omega})+\rho_i^2(e^{2j\omega}+e^{-2j\omega})} + \sigma_v^2 . \tag{57}$$

In this example we chose $\rho_1 = \rho_2 = 0.99$, $f_1 = 0.2H_2$, $f_2 = 0.225$ Hz, $E_1 = E_2 = 1$, $\sigma_v^2 = 2$. Thus, the SNR is -3dB for each of the two narrowband processes. The equivalent ARMA description of this process is

$$\frac{\sigma b(z)}{a(z)} = \frac{1.5856(1-0.8706z^{-1}+1.9194z^{-2}-0.7610z^{-3}+0.7641z^{-4})}{1-0.9217z^{-1}+2.1502z^{-2}-0.9036z^{-3}+0.9606z^{-4}} . \tag{58}$$

The number of data points was chosen to be n=50. Figures 4a and 4b show the exact and the approximate bounds on the spectrum. Note that the peaks of the lower bound are lower than the dip of the upper bounds. This means that with high probability the two narrowband processes cannot be resolved by any unbiased estimator of the spectrum. This phenomenon cannot be predicted by the asymptotic approximation, but only by the exact bound.

Example #5:

This example is similar to the previous one, except that the white noise variance was increased to 4 (i.e., the SNR is -6 dB). The equivalent ARMA description of this process is

$$\frac{\sigma\, b(z)}{a(z)} = \frac{2.1630(1-0.8863z^{-1} + 1.9883z^{-2}-0.8032z^{-3}+0.8212z^{-4})}{1-0.9217z^{-1}+2.1502z^{-2}-0.9306z^{-3}+0.9606z^{-4}} . \tag{59}$$

The exact and approximate spectral bounds are shown in Figures 5a and 5b, for n=50. As we see, the peaks of the lower bounds are much below the dip of the upper bound, is that the two processes are not likely to be resolved at all.

Example #6:

This example is similar to the two previous ones, except that $\sigma_v^2 = 8$, i.e., the SNR is -9 dB. The equivalent ARMA description of this process is

$$\frac{\sigma b(z)}{a(z)} = \frac{2.9798(1-0.8979z^{-1}+2.0407z^{-2}-0.8353z^{-3}+0.8655z^{-4})}{1-0.9217z^{-1}+2.1502z^{-2}-0.9036z^{-3}+0.9606z^{4}} . \tag{60}$$

Figures 6a and 6b show the exact and the asymptotic spectral bounds. Here even the asymptotic approximation indicates that the two processes cannot be resolved. However, the exact bound indicates that with high probability none of processes can be detected.

Example #7:

This example is similar to example #4, except that f_2 was changed to 0.2125 Hz. The equivalent ARMA description of this process is

$$\frac{\sigma b(z)}{a(z)} = \frac{1.5723(1-1.0192z^{-1}+2.0215z^{-2}-0.8985z^{-3}+0.7771z^{-4})}{1-1.0738z^{-1}+2.2438z^{-2}-1.05292^{-3}+0.9606z^{-4}} . \tag{61}$$

Figures 7a and 7b show the exact and the asymptotic spectral bounds. The two narrowband processes are evidently indistinguishable, but the approximate bound fails to indicate this.

Example #8:

This example is similar to examples #4 and #7, except that f_2 was changed to 0.25 Hz. The equivalent ARMA description of this process is

$$\frac{\sigma b(z)}{a(z)} = \frac{1.5943(1-0.5763z^{-1}+1.7395z^{-2}-0.5011z^{-3}+0.7558z^{-4})}{1-0.6119z^{-1}+1.9603z^{-2}-0.5997z^{-3}+0.9606z^{-4}} . \qquad (62)$$

Figures 8a and 8b show the exact and the asymptotic spectral bounds. Now the two frequencies are sufficiently for apart, so that the two bounds are similar and both indicate that the two can be easily resolved.

6. DISCUSSION

We presented an algorithm for computing the exact Fisher information matrix of parametric Gaussian time series whose random components are stationary. The algorithm is computationally efficient, requiring a number of operations proportional to N^2 for computing the matrices $\{J_n(\theta),\ 1 \le n \le N\}$. The Cramer-Rao bound for unbiased estimates of the parameters is simply the inverse of the information matrix.

The algorithm was specialized to the case of ARMA processes with additive white noise, and closed form expressions were derived for the covariances and their partial derivatives. Some common nonstationary time series can be similarly handled, such as sums of sinusoids in white or colored noise, rational impulse responses in white or colored noise, etc.

Examination of the exact information matrix of ARMA processes reveals an interesting fact. As is well known, the asymptotic information matrix of ARMA processes is symmetric in the numerator and denominator parameters. In other words, interchanging the numerator and the denominator polynomials leaves the information matrix unchanged, except for row and column permutations [1, pp. 240]. However, the exact information matrix does not share this symmetry property. See for example the difference between the denominator and the numerator parameters in Figure 1a, when the number of data is small. This observation offers a partial explanation to the well known fact that with a small number of data points it is much more difficult to accurately estimate zeroes than poles.

We finally note that the Cramer-Rao bound for short data records is not necessarily tight, i.e., efficient ARMA algorithms may not exist. However, the CRB still provides a lower bound on the performance of any given algorithm. We should stress that the bound applies to unbiased estimates

only. While there is no evidence that there exist unbiased ARMA estimation methods,. most existing algorithms are designated to be approximately unbiased. For such algorithms, the inverse of the information matrix offers a reasonable measure of achievable performance.

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APPENDIX A: PROOF OF THE INFOMRATION MATRIX FORMULA

The algorithm of the joint density function $f(\underline{y})$ given in (2) is

$$\log f(\underline{y}) = -\frac{N}{2}\log 2\pi - \frac{1}{2}\log\det R(\theta) - \frac{1}{2}[\underline{y}-\underline{m}(\theta)]^T R^{-1}(\theta)[\underline{y}-\underline{m}(\theta)] \quad (A.1)$$

Differentiation with respect to θ_k yields

$$\frac{\partial \log f(\underline{y})}{\partial\theta_k} = -\frac{1}{2}\operatorname{tr}\{R^{-1}(\theta)\frac{\partial R(\theta)}{\partial\theta_k}\} + \frac{1}{2}[\underline{y}-\underline{m}(\theta)]^T R^{-1}(\theta)\frac{\partial R(\theta)}{\partial\theta_k}R^{-1}(\theta)[\underline{y}-\underline{m}(\theta)]$$

$$+ [\frac{\partial \underline{m}(\theta)}{\partial\theta_k}]^T R^{-1}(\theta)[\underline{y}-\underline{m}(\theta)] . \quad (A.2)$$

Multiplying $\partial\log f(\underline{y})/\partial\theta_k$ by $\partial\log f(\underline{y})/\partial\theta_\ell$ and taking expected value yields

$$[J(\theta)]_{k,\ell} = -\frac{1}{4}\operatorname{tr}\{R^{-1}(\theta)\frac{\partial R(\theta)}{\partial\theta_k}\}\operatorname{tr}\{R^{-1}(\theta)\frac{\partial R(\theta)}{\partial\theta_\ell}\} + [\frac{\partial\underline{m}(\theta)}{\partial\theta_k}]^T R^{-1}(\theta)[\frac{\partial\underline{m}(\theta)}{\partial\theta_\ell}]$$

$$+ \frac{1}{4}E\{[\underline{y}-\underline{m}(\theta)]^T R^{-1}(\theta)[\underline{y}-\underline{m}(\theta)]\frac{\partial R(\theta)}{\partial\theta_k}R^{-1}(\theta)[\underline{y}-\underline{m}(\theta)]^T R^{-1}(\theta)\frac{\partial R(\theta)}{\partial\theta_\ell}R^{-1}(\theta)[\underline{y}-\underline{m}(\theta)]\} .$$

$$(A.3)$$

To evaluate the last term let us denote

$$\underline{x} = \underline{y} - \underline{m}(\theta) \quad ; \; A = R^{-1}(\theta)\frac{\partial R(\theta)}{\partial\theta_k}R^{-1}(\theta) \quad ; \; B = R^{-1}(\theta)\frac{\partial R(\theta)}{\partial\theta_\ell}R^{-1}(\theta) . \quad (A.4)$$

Then

$$E\{\underline{x}^T A\underline{x}\,\underline{x}^T B\,\underline{x}\} = E\{\sum_{i,j,m,n} x_i A_{ij} x_j x_m B_{mn} x_n\} = \sum_{i,j,m,n} A_{ij}B_{mn}E\{x_i x_j x_m x_n\}$$

$$= \sum_{i,j,m,n} A_{ij}B_{mn}(R_{ij}R_{mn} + R_{im}R_{jn} + R_{in}R_{jm})$$

$$= (\sum_{i,j} A_{ij}R_{ij})\cdot(\sum_{m,n} B_{mn}R_{mn}) + \sum_{i,j,m,n} A_{ij}R_{jn}B_{nm}R_{mi} + \sum_{i,j,m,n} A_{ij}R_{jm}B_{mn}R_{ni}$$

$$= \operatorname{tr}\{AR\}\cdot\operatorname{tr}\{BR\} + 2\operatorname{tr}\{ARBR\}$$

$$= \operatorname{tr}\{R^{-1}(\theta)\frac{\partial R(\theta)}{\partial\theta_k}\}\cdot\operatorname{tr}\{R^{-1}(\theta)\frac{\partial R(\theta)}{\partial\theta_\ell}\} + 2\operatorname{tr}\{R^{-1}(\theta)\frac{\partial R(\theta)}{\partial\theta_k}R^{-1}(\theta)\frac{\partial R(\theta)}{\partial\theta_\ell}\} .$$

$$(A.5)$$

Substituting (A.5) into (A.3) we finally get

$$[J(\theta)]_{k,\ell} = \frac{1}{2}\mathrm{tr}\{R^{-1}(\theta)\ \frac{\partial R(\theta)}{\partial \theta_k}\ R^{-1}(\theta)\ \frac{\partial \underline{m}(\theta)}{\partial \theta_k}\} + [\frac{\partial \underline{m}(\theta)}{\partial \theta_k}]^T R^{-1}(\theta)[\frac{\partial \underline{m}(\theta)}{\partial \theta_\ell}] \ . \tag{A.6}$$

APPENDIX B: SUMMARY OF THE ALGORITHM

Inputs:

N: number of data points

$\{r_0, \ldots, r_{N-1}\}$: covariances of the given process

$\{s_{0,k}, \ldots, s_{N-1,k},\ 1 < k < M\}$: partial derivatives of the covariances

$\{u_{0,k}, \ldots, u_{N-1,k},\ 1 < k < M\}$: partial derivatives of the mean vector

Initialization:

$\alpha_0 = 1; \quad \alpha_n = 0 \qquad , n=1,\ldots,N-1;$

$\delta = r_0;$

$\eta_{0,k} = s_{0,k} \qquad , k=1,\ldots,M;$

$\xi_{0,k} = s_{0,k}/r_0 \qquad , k=1,\ldots,M;$

$J_{1,k,\ell} = \frac{1}{2}\, s_{0,k} s_{0,\ell}/r_0^2 \qquad , k,\ \ell = 1,\ldots,M;$

Do for n=1,...,N-1:

$$c = \Big(\sum_{i=0}^{n-1} \alpha_i r_{n-i}\Big)/\delta \ ,$$

$$\alpha_k = \sum_{i=0}^{n-1} \alpha_i s_{n-i,k} \qquad , \quad k=1,\ldots,M;$$

$$\delta = \delta \cdot (1-c^2) \ ;$$

$$\eta_{n,k} = d_k \qquad , k=1,\ldots,M;$$

$$t_i = \alpha_i - c\alpha_{n-i} \qquad , i=0,\ldots,n;$$

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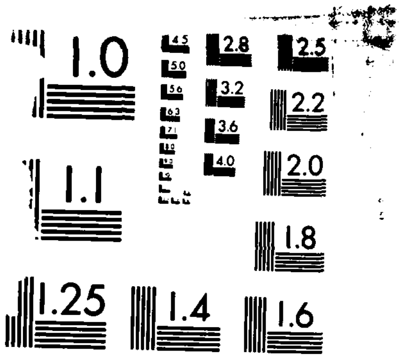


$$\alpha_i = t_i \qquad , \; t=0,\dots,n;$$

$$e_k = \sum_{i=0}^{n} \eta_{i,k}\alpha_{n-i} \qquad , \; k=1,\dots,M;$$

$$\left.\begin{array}{ll} t_i = \eta_{i,k} - c\eta_{n-i,k} & , \; i=0,\dots,n\,; \\ \eta_{i,k} = t_i & , \; i=0,\dots,n; \end{array}\right\} k=1,\dots,M$$

$$\left.\begin{array}{ll} t_i = \xi_{i,k} - c\xi_{n-i,k} + e_k(\alpha_{n-i} - c\alpha_i)/\delta & , \; i = 0,\dots,n\,; \\ \xi_{i,k} = t_i & , \; i=0,\dots,n\,; \end{array}\right\} k = 1,\dots,M$$

$$f_k = \sum_{i=0}^{n} \alpha_i \eta_{i,k} \qquad , \; k=1,\dots,M \;;$$

$$g_k = \sum_{i=0}^{n} \alpha_{n-i}\, u_{i,k} \qquad , \; k=1,\dots,M \;;$$

$$J_{n+1,k,\ell} = J_{n,k,\ell} + \Big(\sum_{i=0}^{n} \eta_{i,k}\xi_{i,\ell}\Big)/\delta - \frac{1}{2} f_k f_\ell/\delta^2 + g_k g_\ell/\delta \qquad , \; k,\ \ell=1,\dots,M \;;$$

Comments:

(i) The vector $\underline{\alpha}$, $\underline{\eta}_k$, $\underline{\xi}_k$ and the scalar δ are overwritten at each step by the new values. This helps keeps storage requirements proportional to N, rather than N^2. The temporary storage vector $\underline{t}$ is used in updating the vectors $\underline{\alpha}$, $\underline{\eta}_k$, $\underline{\xi}_k$.

(ii) The Fisher information matrix is not overwritten at each step; however, the algorithm can be easily modified by letting the new value of J overwrite the old value.

(iii) The update of $\underline{\eta}_k$ is split into two steps. First d_k is added to the n-th component, and the partial result is used to compute e_k (cf. (27)). Then the updating of $\underline{\eta}_k$ is completed.

FIGURE CAPTIONS

Figure 1: Example 1-- a) The asymptotic and exact Cramer-Rao bounds for the ARMA parameter estimates, b) the exact Cramer-Rao bound for the spectrum, c) the asymptotic Cramer-Rao bound for the log spectrum

Figure 2: Example 2-- a) The asymptotic and exact Cramer-Rao bounds for the ARMA parameter estimates, b) the exact Cramer-Rao bound for the spectrum, c) the asymptotic Cramer-Rao bound for the log spectrum

Figure 3: Example 3-- a) The asymptotic and exact Cramer-Rao bounds for the ARMA parameter estimates, b) the exact Cramer-Rao bound for the spectrum, c) the asymptotic Cramer-Rao bound for the log spectrum

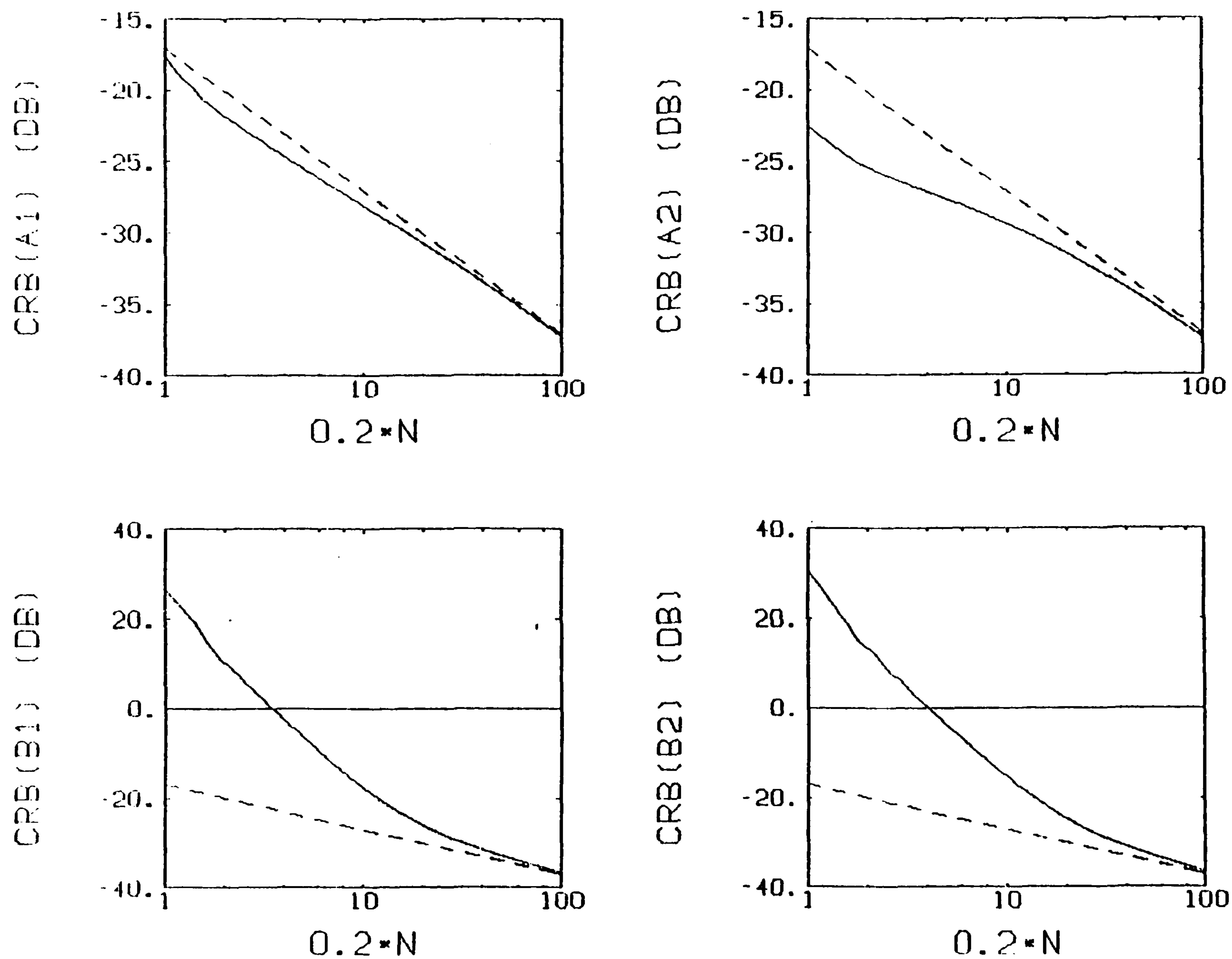
Figure 4: Example 4-- a) the exact Cramer Rao bound for the log spectrum, b) the asymptotic Cramer-Rao bound for the log spectrum

Figure 5: Example 5-- a) the exact Cramer Rao bound for the log spectrum, b) the asymptotic Cramer-Rao bound for the log spectrum

Figure 6: Example 6-- a) the exact Cramer Rao bound for the log spectrum, b) the asymptotic Cramer-Rao bound for the log spectrum

Figure 7: Example 7-- a) the exact Cramer Rao bound for the log spectrum, b) the asymptotic Cramer-Rao bound for the log spectrum

Figure 8: Example 8-- a) the exact Cramer Rao bound for the log spectrum, b) the asymptotic Cramer-Rao bound for the log spectrum



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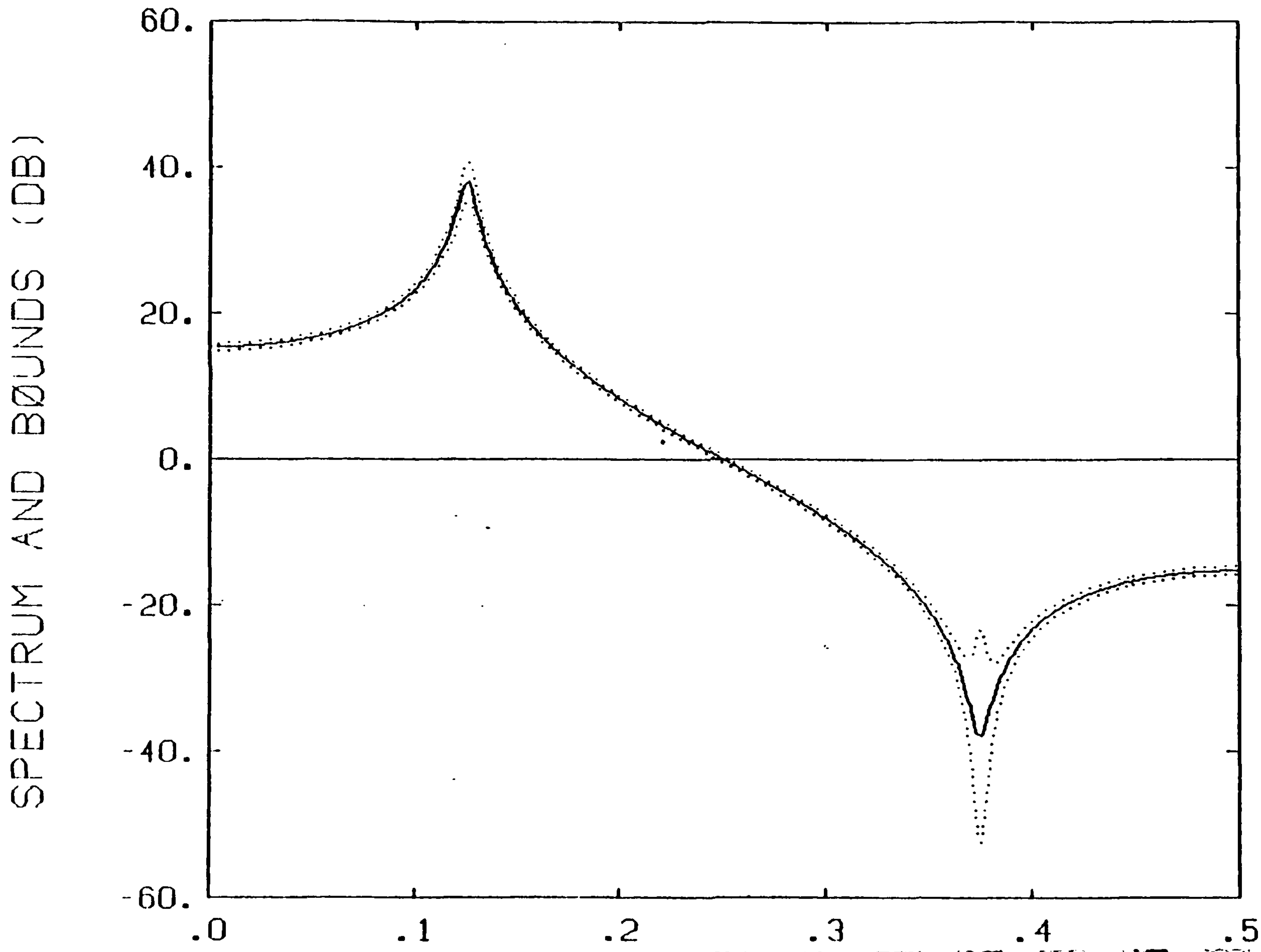
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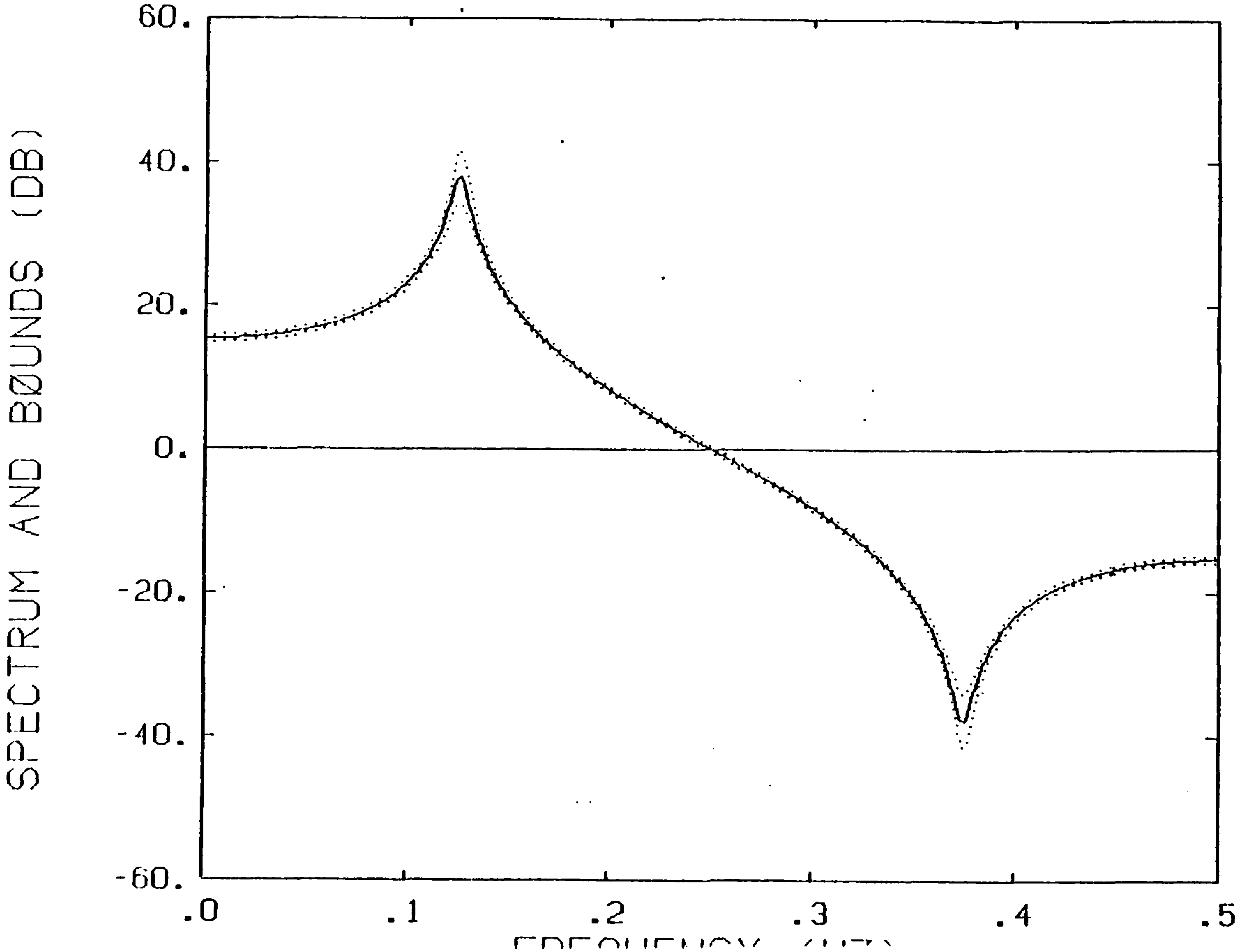
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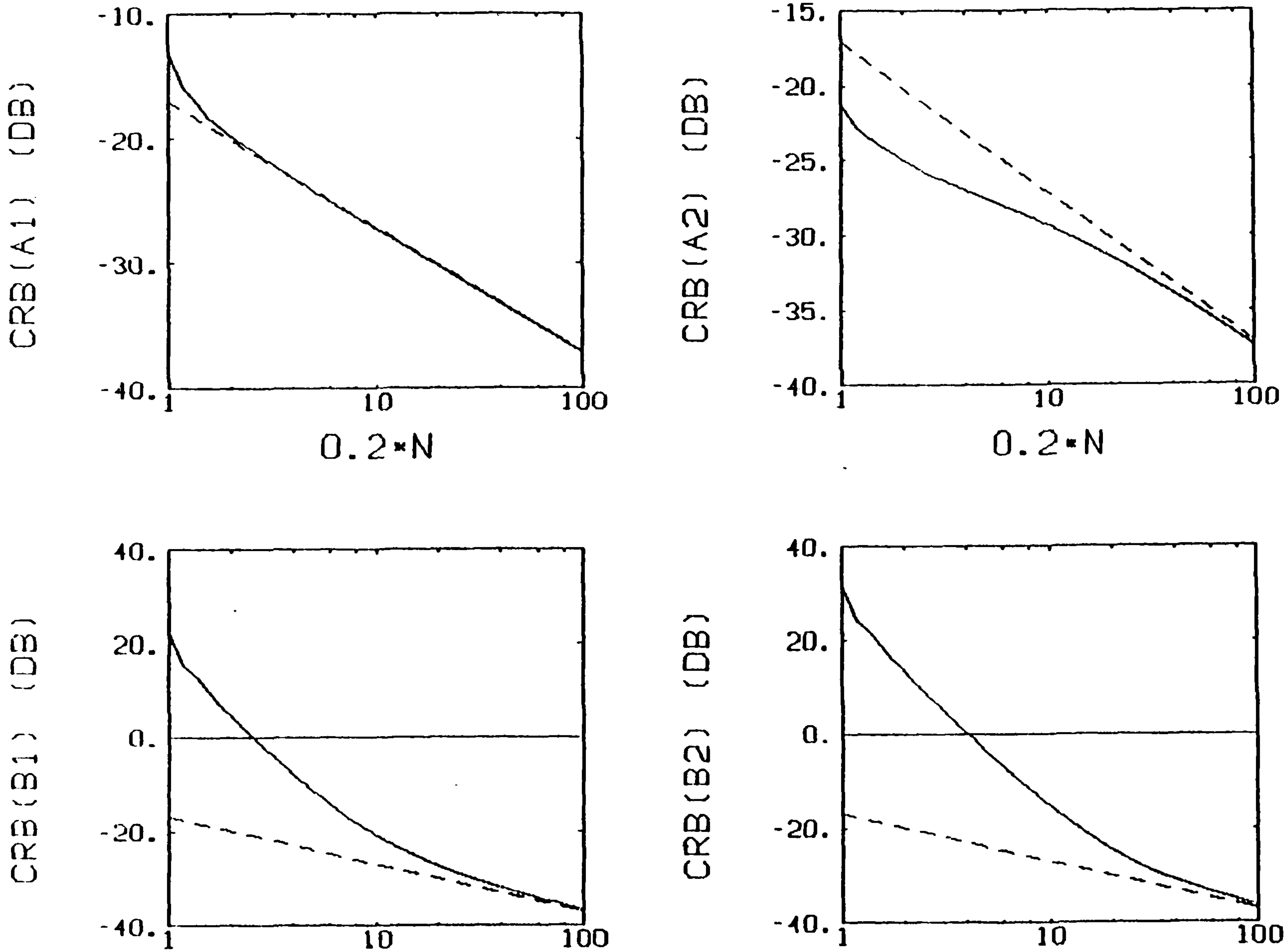
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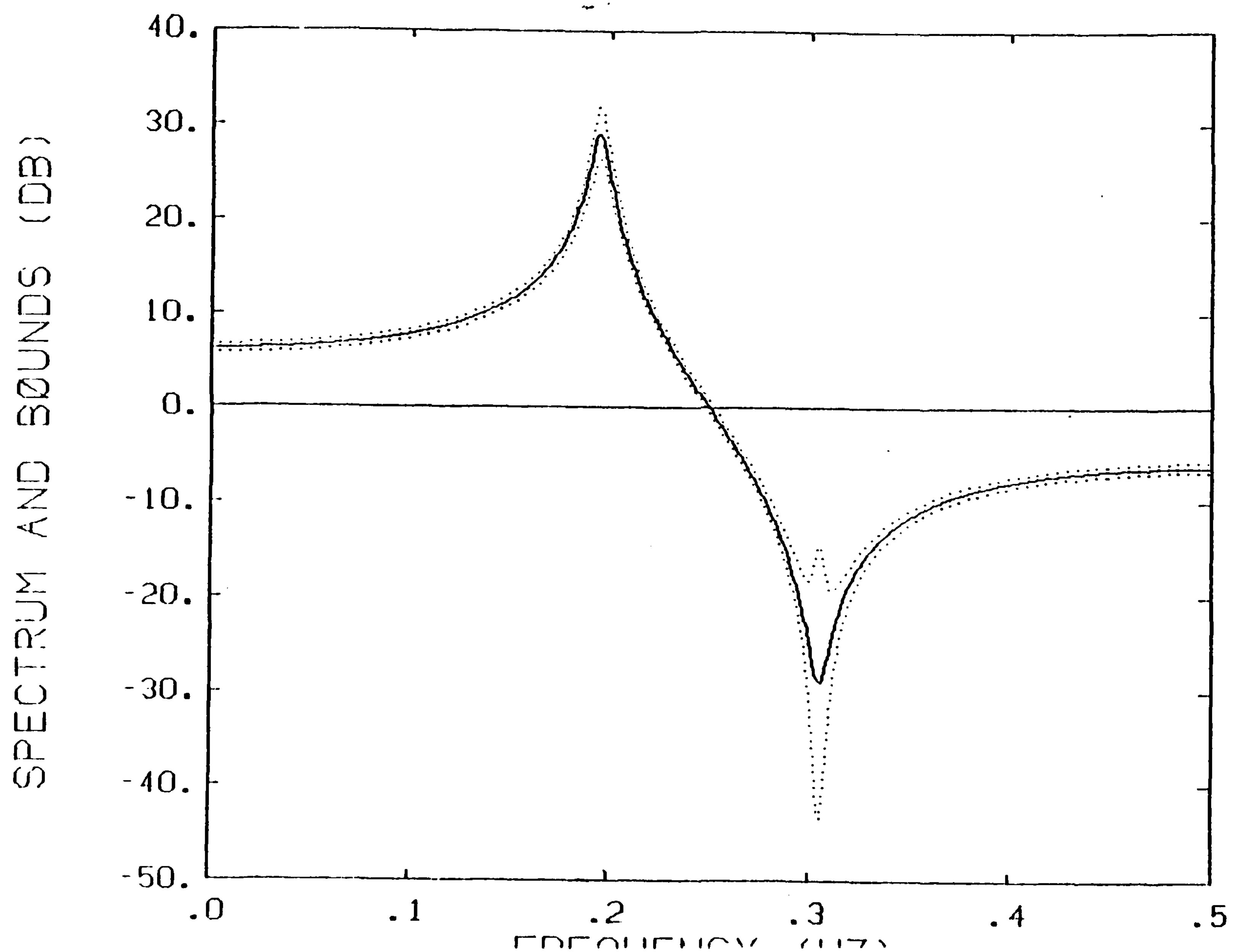

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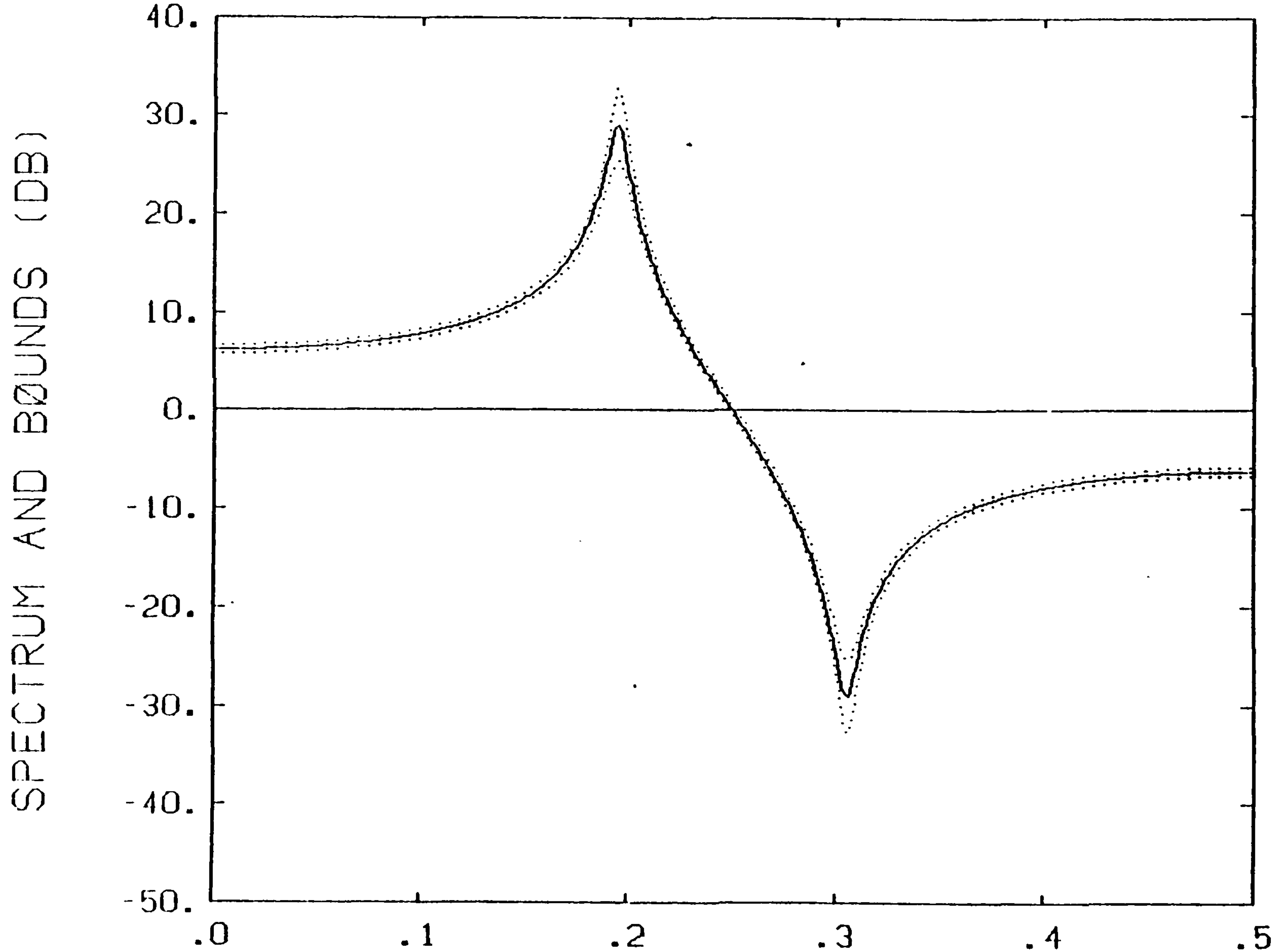

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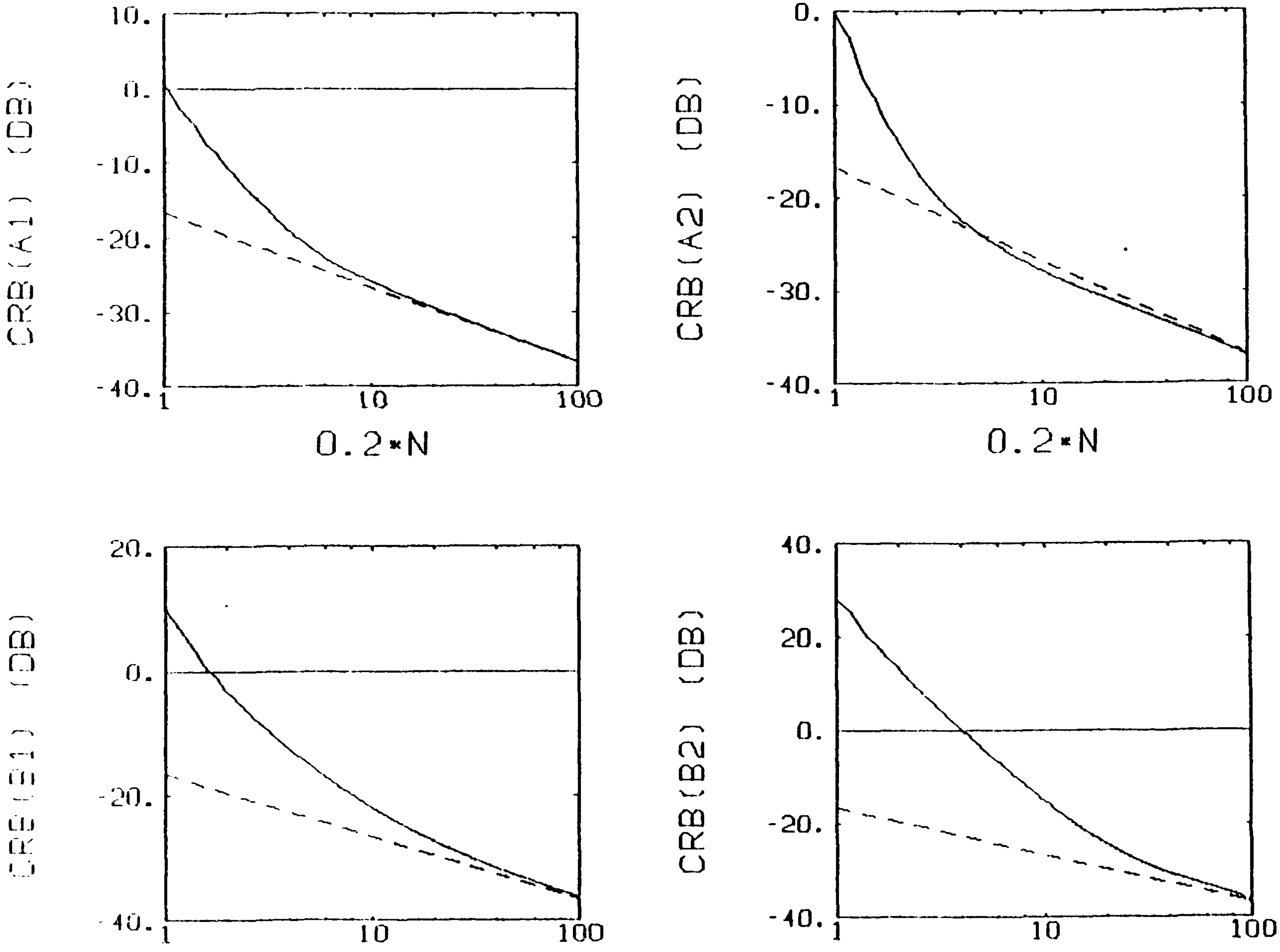

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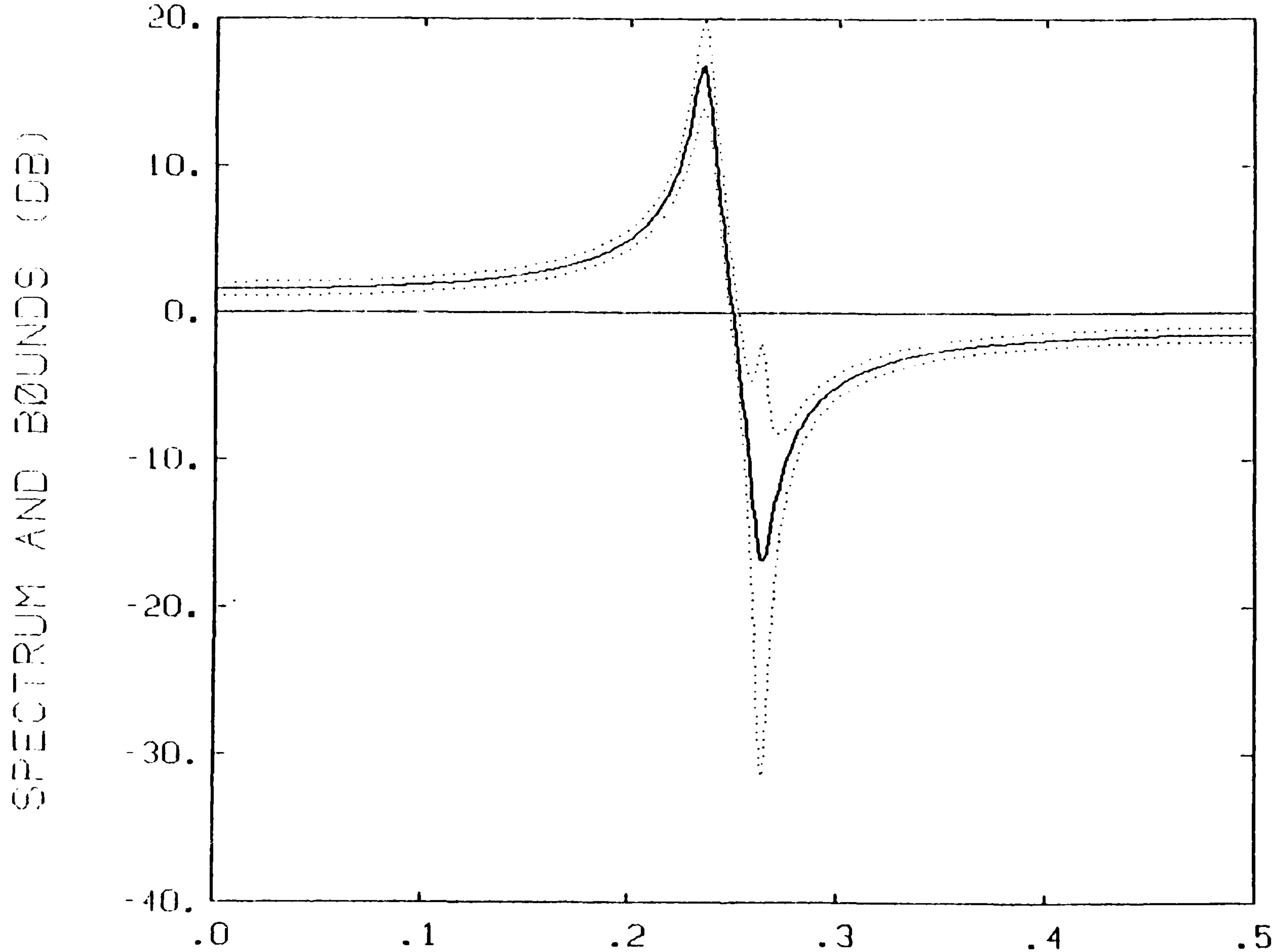

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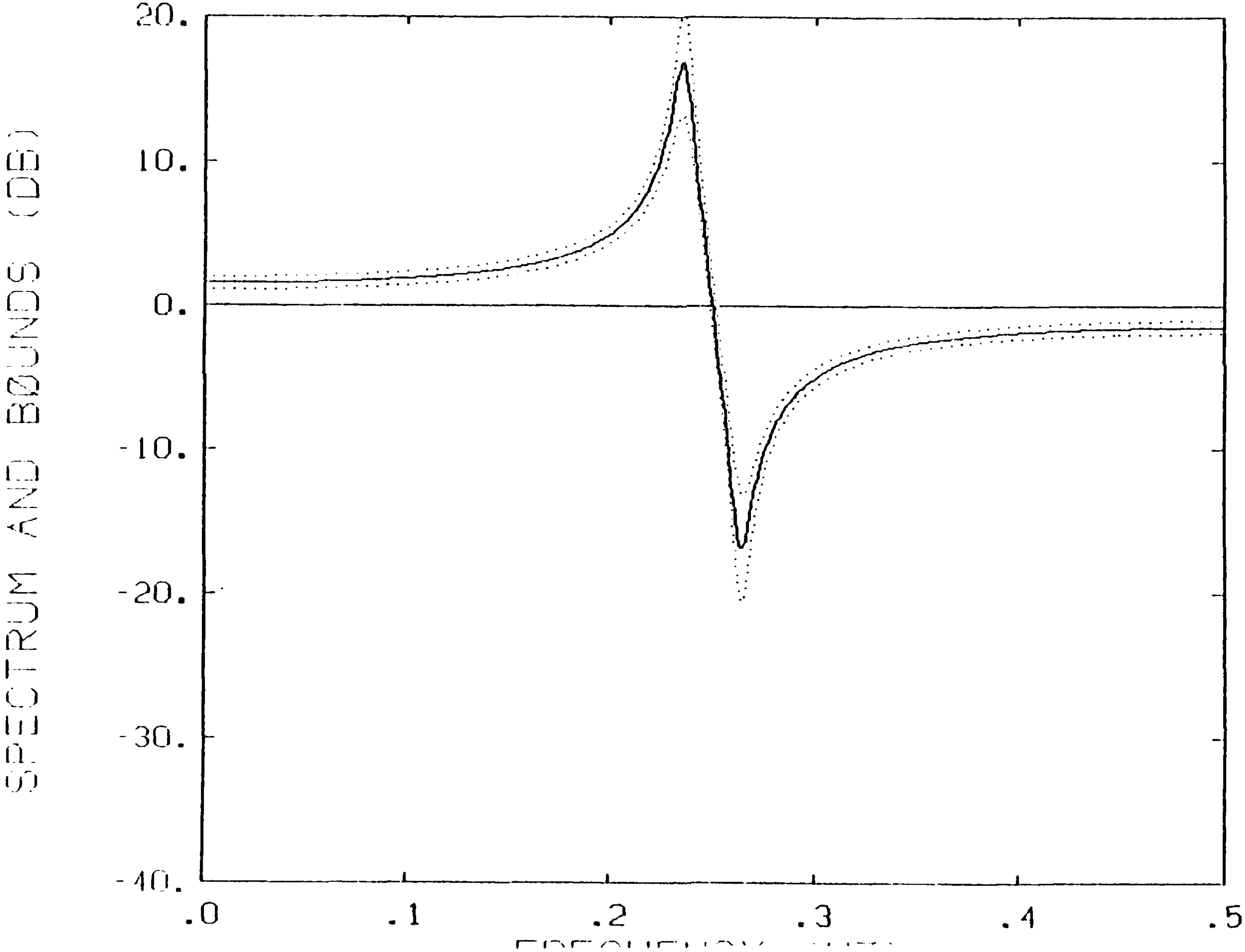

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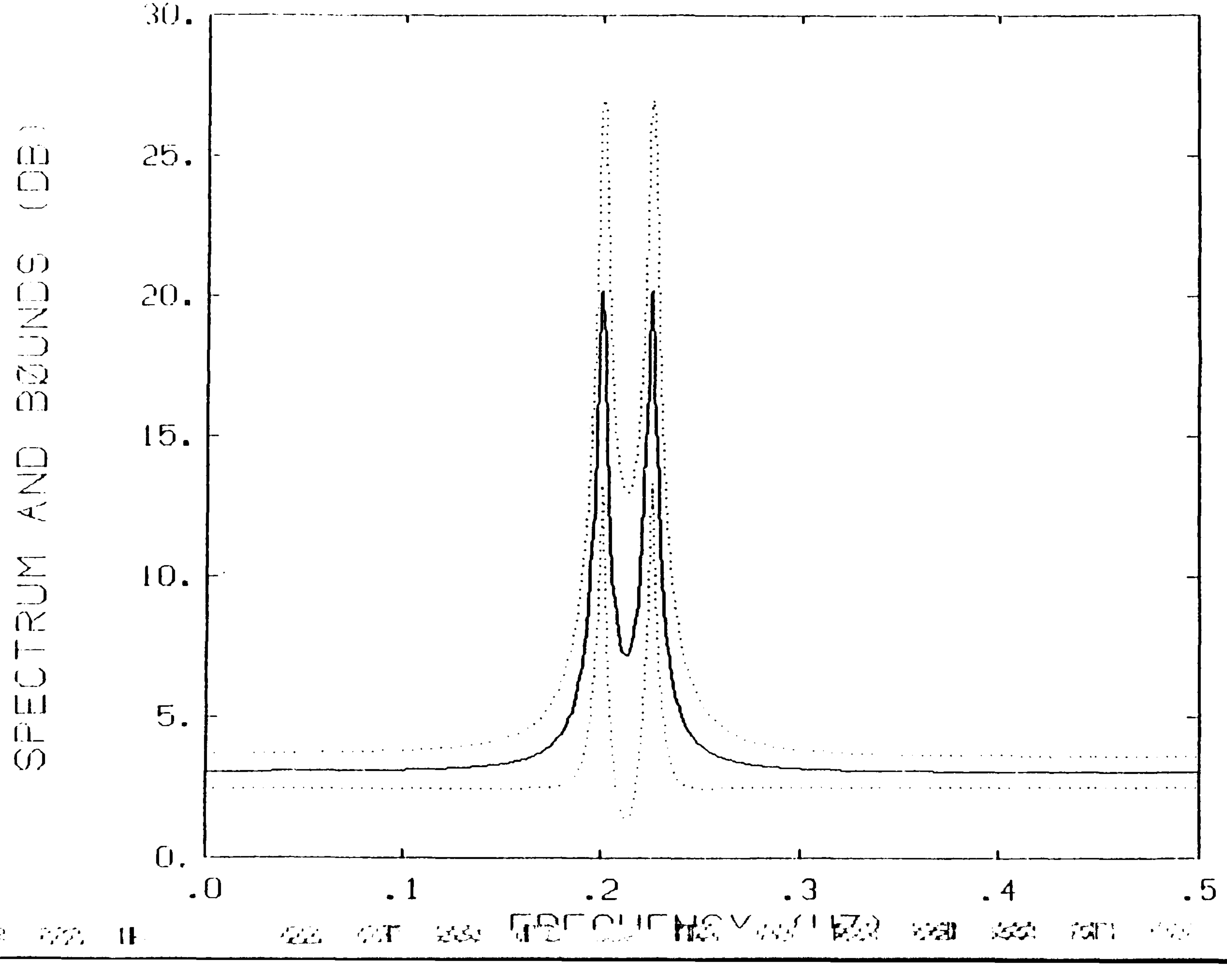
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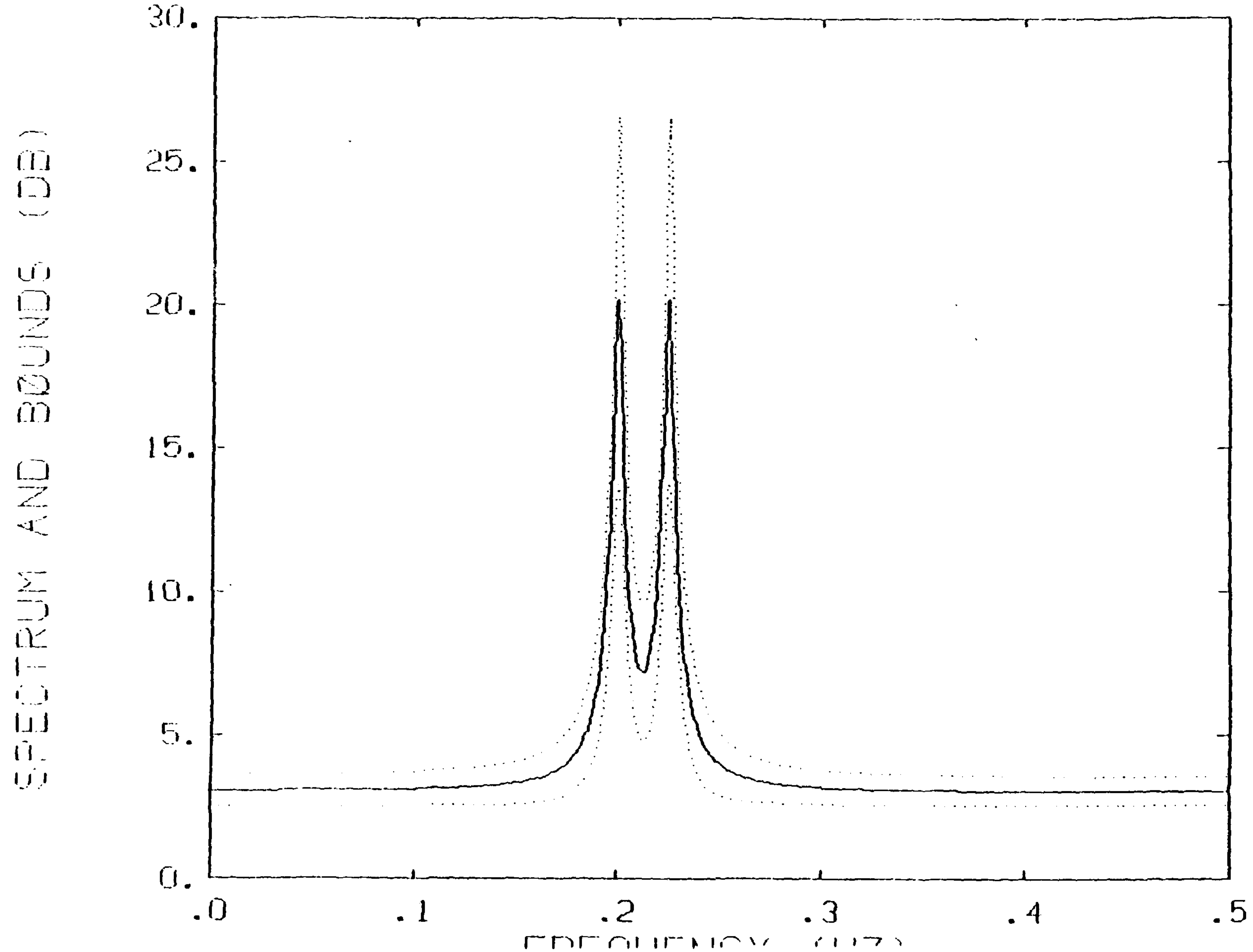
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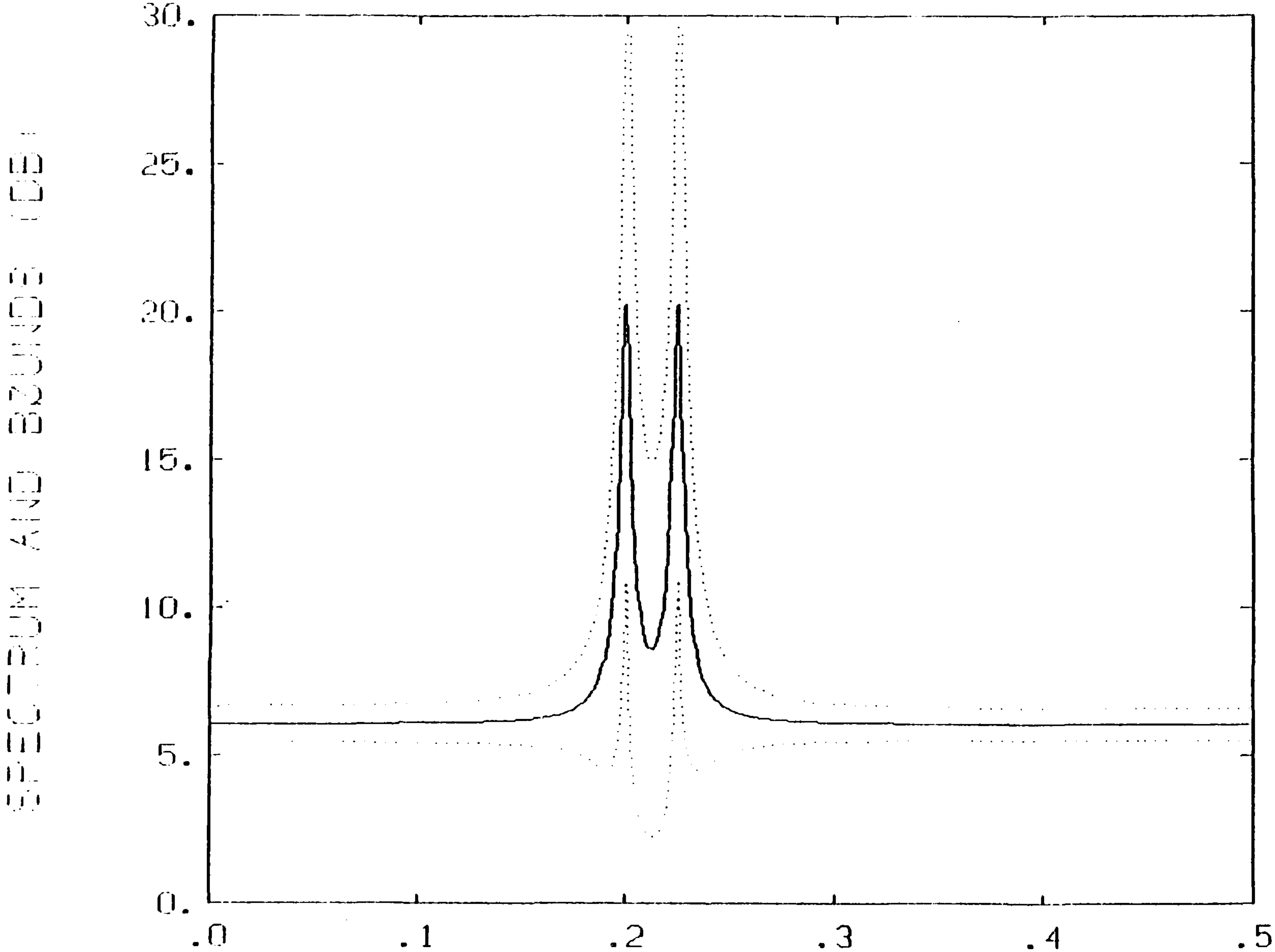

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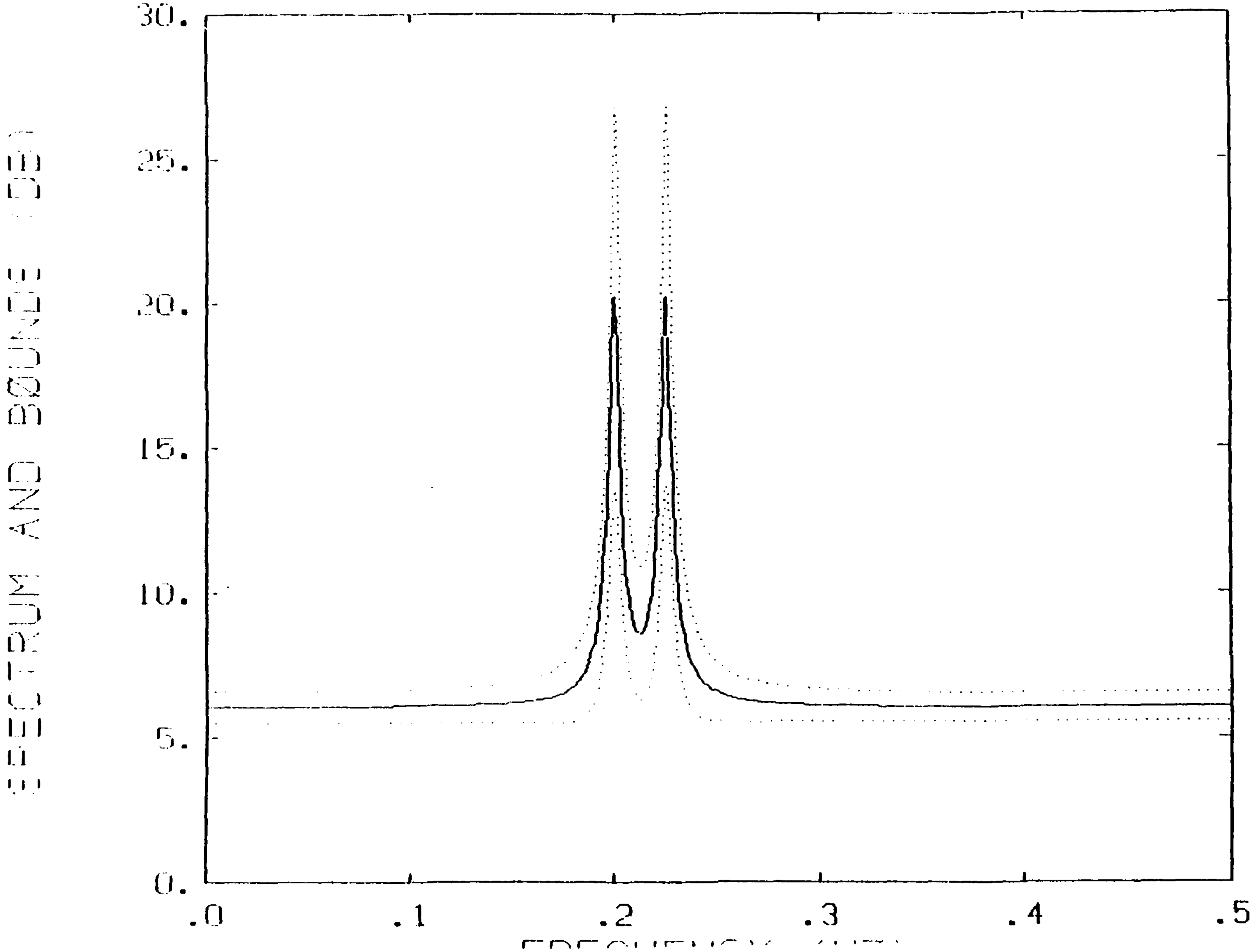

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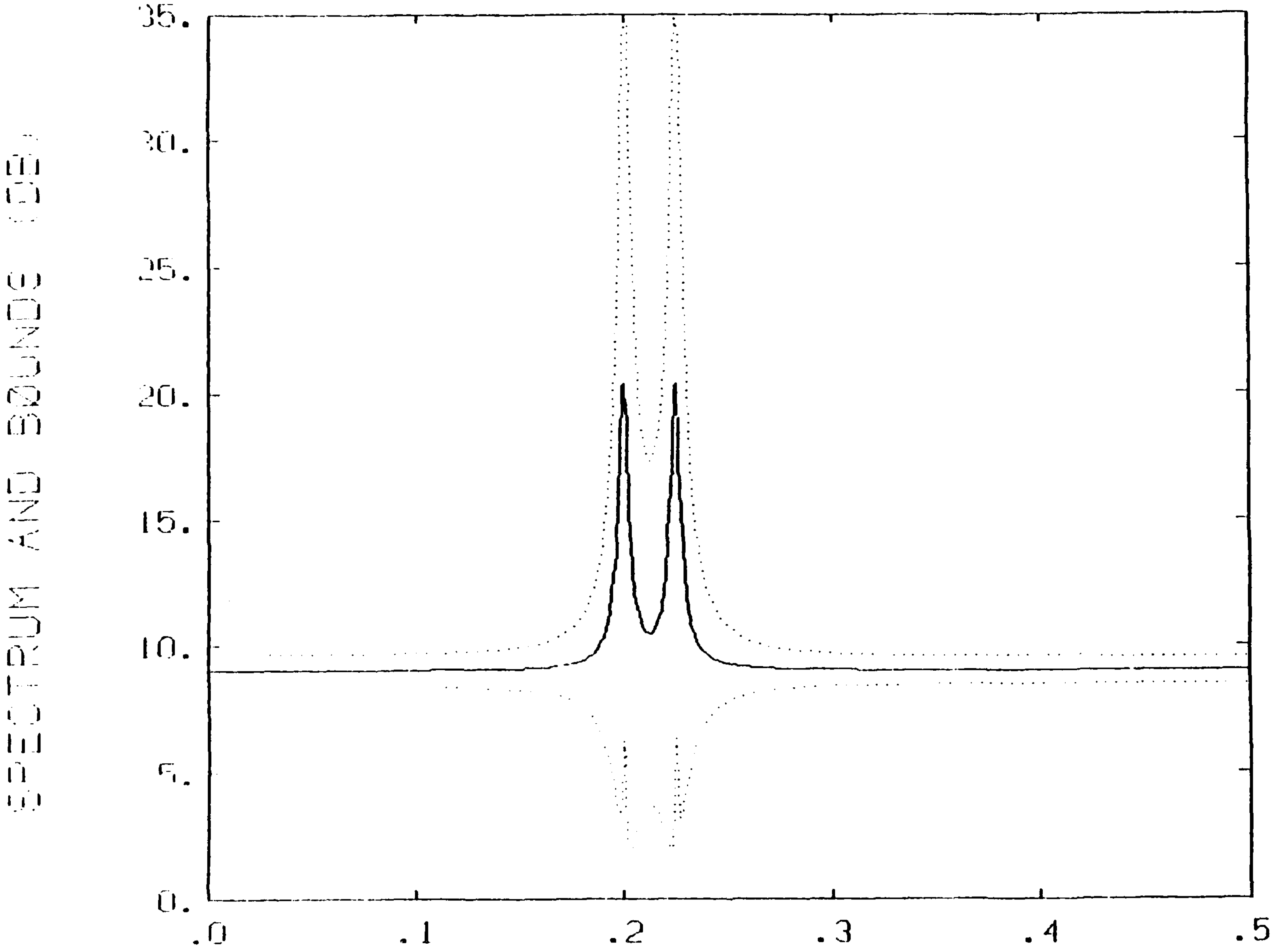

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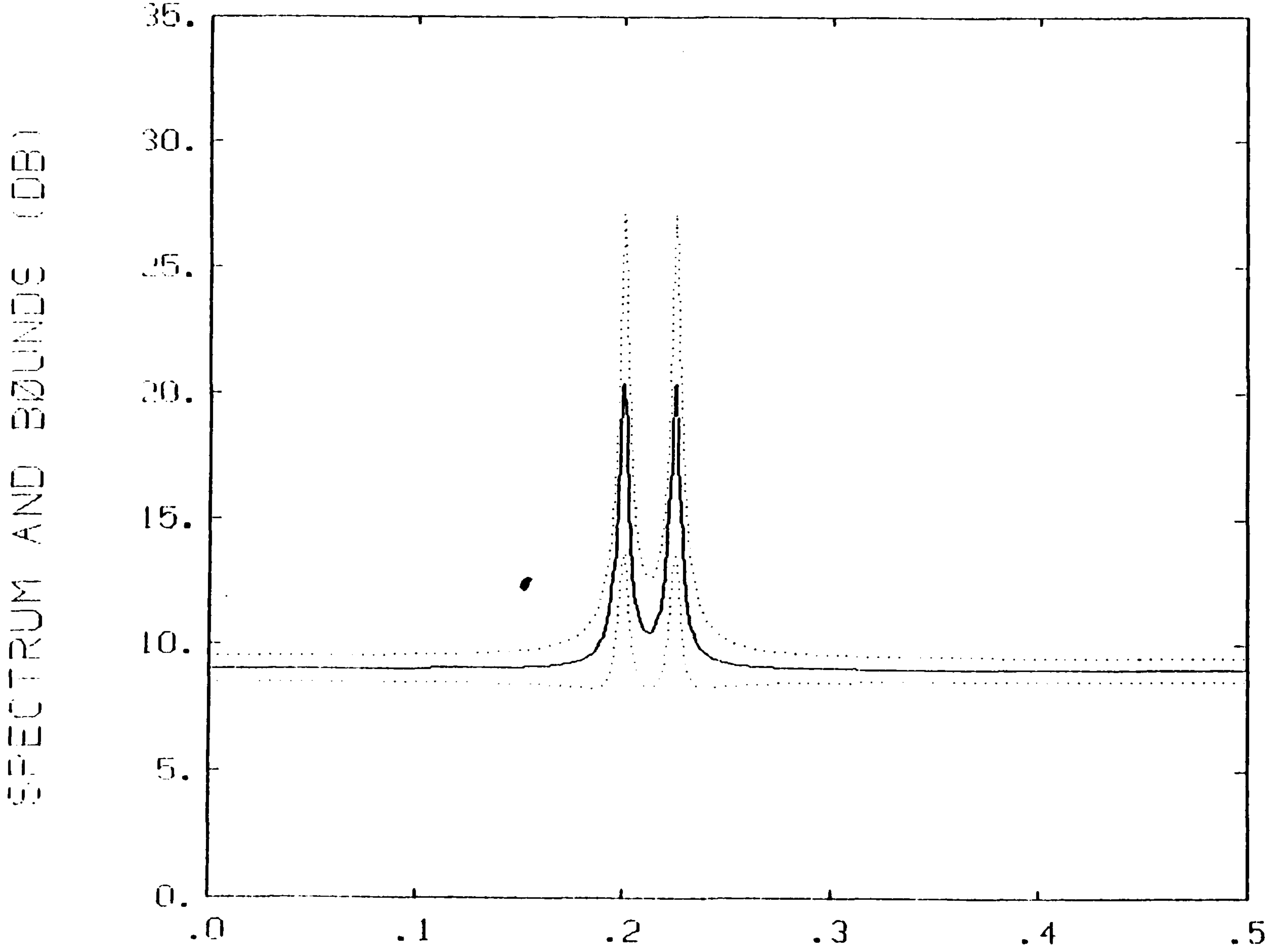

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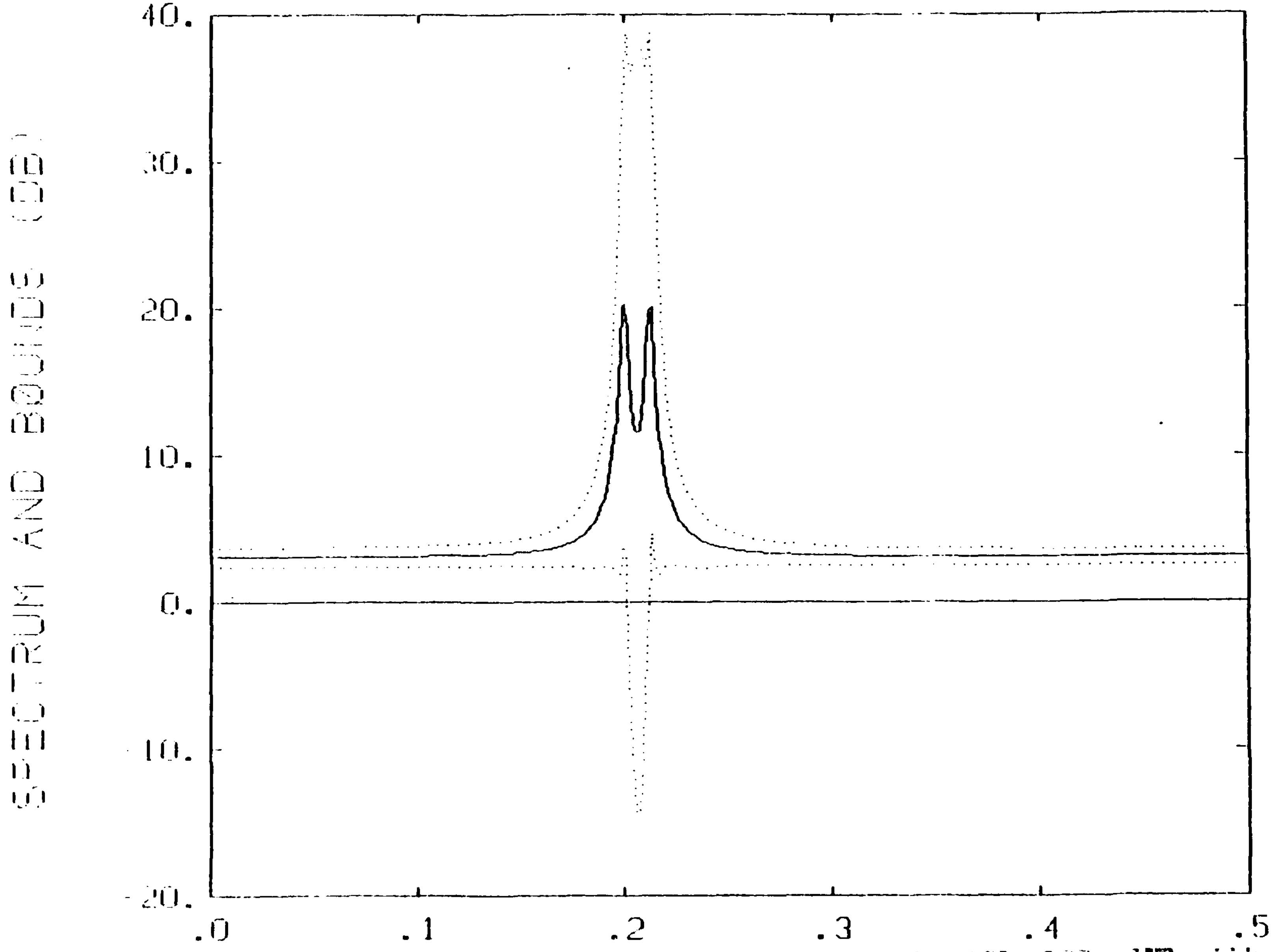
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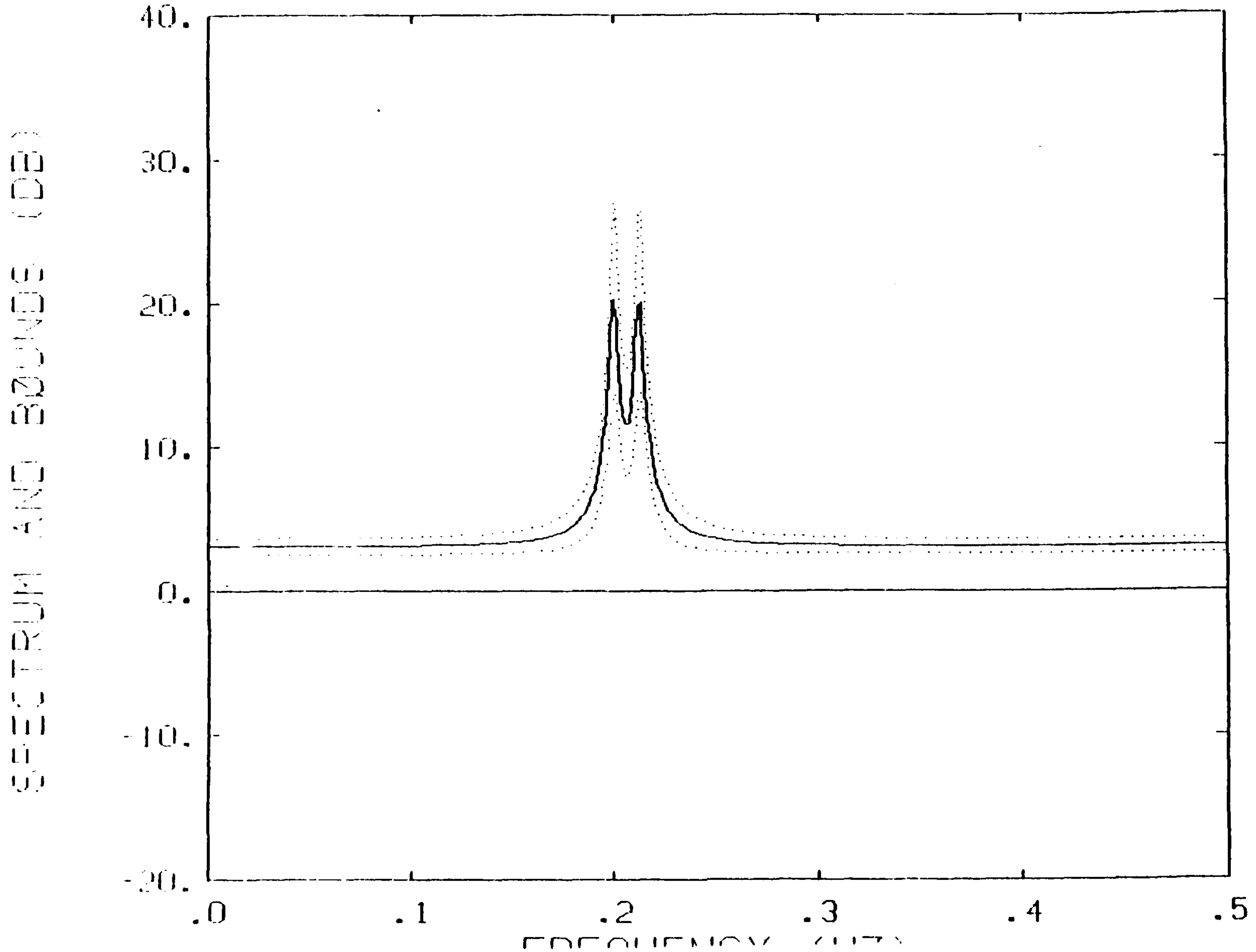
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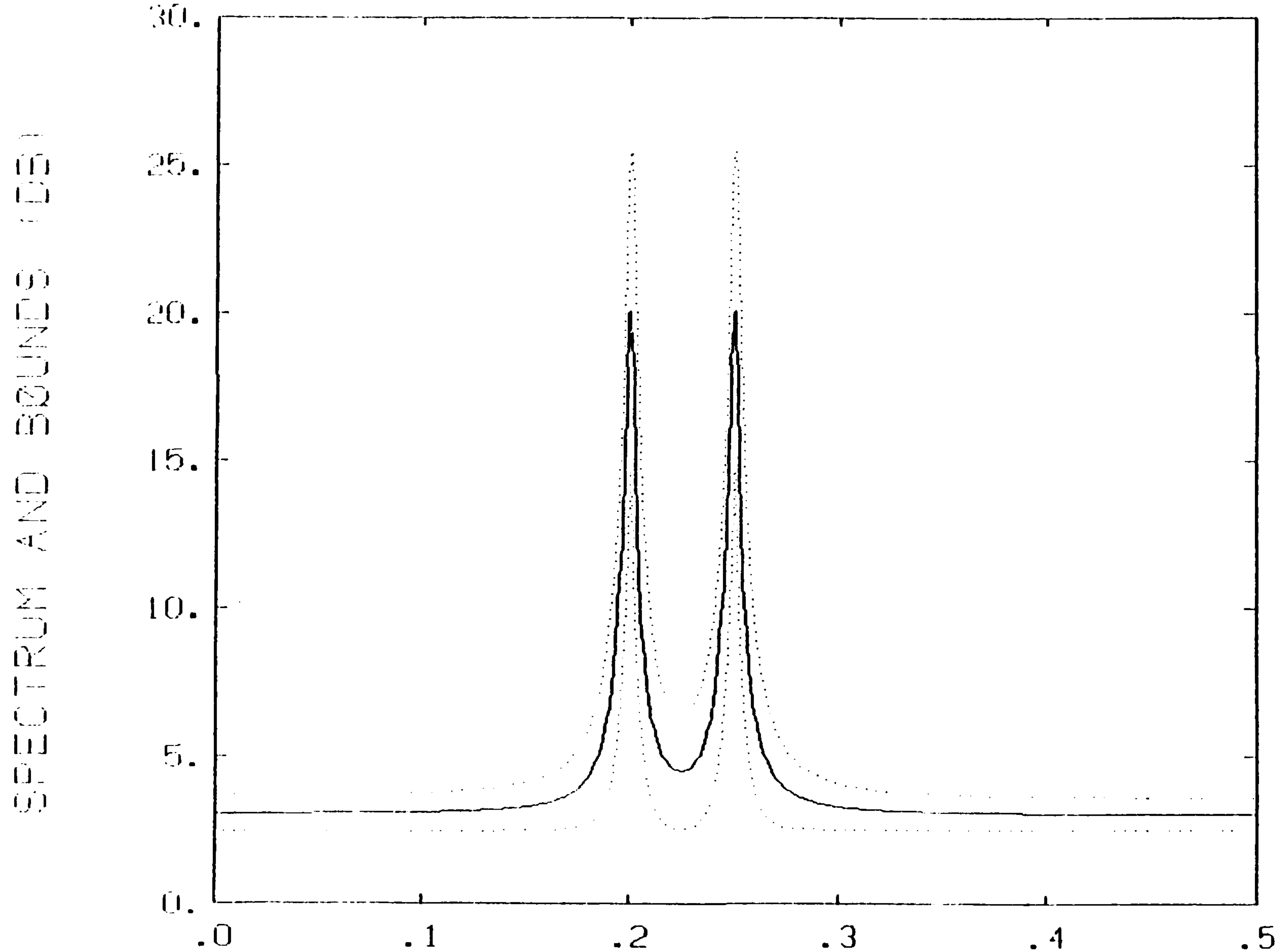

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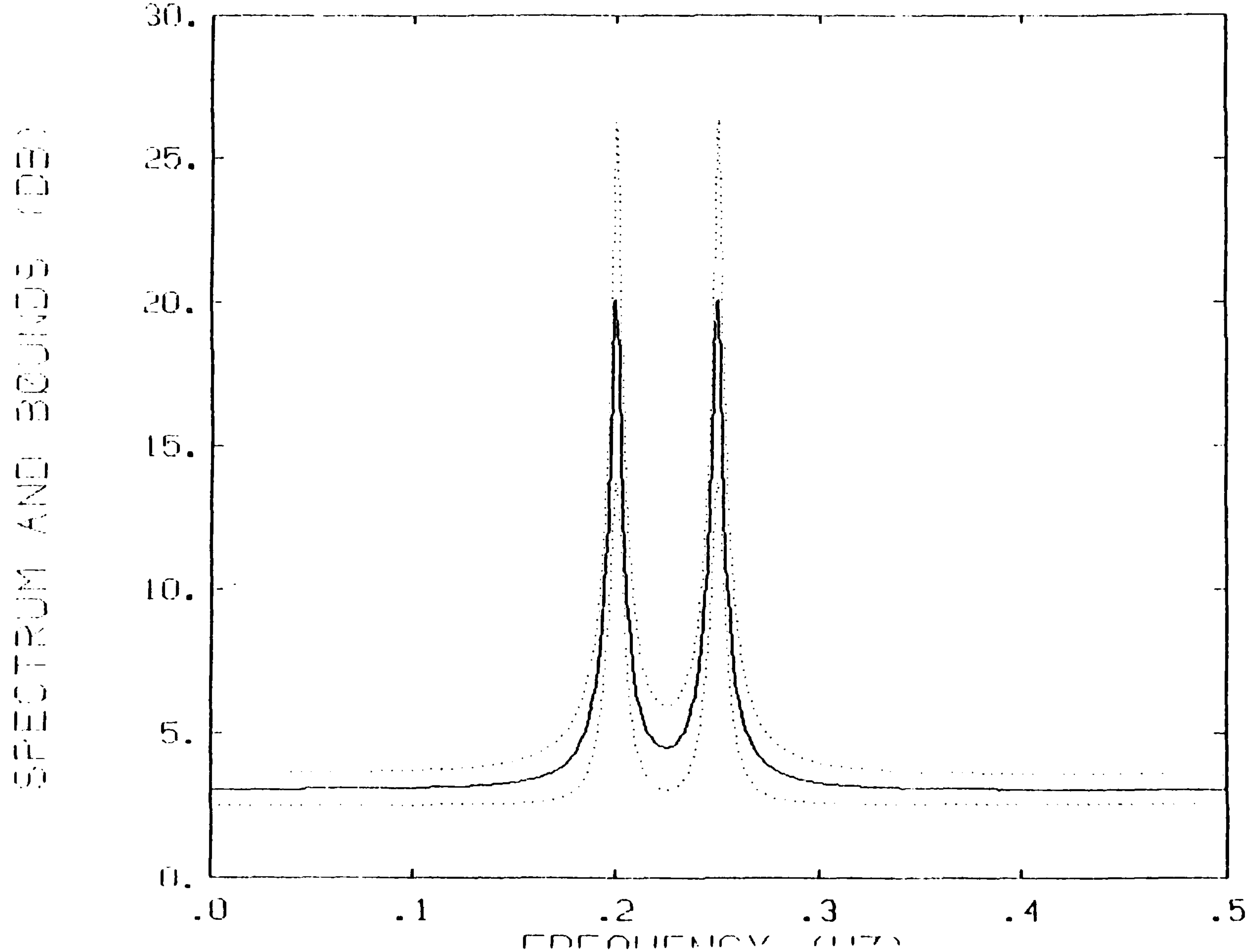

APPENDIX E

MULTICHANNEL ARMA SPECTRAL ESTIMATION BY THE MODIFIED YULE-WALKER METHOD

MULTICHANNEL ARMA SPECTRAL ESTIMATION BY THE MODIFIED YULE-WALKER METHOD

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ABSTRACT

This paper proposes an algorithm for estimating the power spectra of multichannel wide-sense stationary processes. The processes are modeled as the output of a multivariable linear system driven by white noise. The transfer function of the system is given by a numerator matrix of polynomials divided by a scalar denominator polynomial. The denominator polynomial is estimated first, using the overdetermined, order over-estimated, modified Yule-Walker method. Modal decomposition is used to eliminate superfluous modes to reduce the order of the transfer function. Finally, the numerator of the spectral density matrix is estimated.

This work was supported by the Army Research Office under contract no. DAAG29-83-C-0027.

1. INTRODUCTION

Parametric models are widely used in the statistical analysis of scalar time series. In particular, autoregressive (AR) and autoregressive moving-averge (ARMA) modeling has proven to be very successful in many applications [1],[2]. Many problems of practical interest involve vector processes. As examples we mention signals in an acoustic and seismic arrays. It often happens in such applications that important information is present in the cospectra of the various channels (rather than in the autospectra). In such applications it is necessary to perform multichannel processing in order to extract the desired information.

Traditional multichannel time series analysis is based on the use of periodograms and windowed periodograms [3]. Multichannel maximum entropy spectral analysis has also gained some popularity in recent years [4]. Parametric modeling for multichannel time series was discussed by several authors [3],[5],[6]. Usually, the multichannel ARMA model, which is a special case of left matrix fraction description [7], is used in these discussions.

The main problem in using parametric models for multichannel time series is their high dimensionality. The number of free parameters is generally proportional to the square of the number of channels. Note that even a relatively simple problem involving a two-channel ARMA model of order (2,2) has 20 free parameters. Simultaneous estimation of so many parameters using a maximum likelihood method is difficult. Problems such as obtaining initial conditions, searching among multiple local minima and selecting the appropriate order are extremely difficult to handle.

In this paper we propose a parametric spectral estimation algorithm which is aimed at circumventing some of the practical difficulties encountered in maximum likelihood estimation. The algorithm uses the sample covariances

(rather than the data directly), and is an extension of the scalar modified Yule-Walker (MYW) method with modal decomposition, reported in [8]. The proposed technique is non-iterative and for the most part requires the solution of sets of linear equations. The parameters of the denominator and numerator of the spectral density matrix are estimated in two separate steps. This alleviates somewhat the problem of high dimensionality.

While the proposed estimation procedure is not asymptotically efficient, it appears to be more robust and considerably less complex (in terms of computational requirements) than the maximum likelihood estimator. The MYW based approach seems, therefore, better suited for practical spectral analysis problems than the maximum likelihood approach. The facts that initial conditions are not required and that the computations consist largely of linear least-squares fits, makes the proposed approach especially attractive.

The outline of the paper is as follows. In section 2 we present the model to be used, and introduce some basic notations. In section 3 we give a detailed description of the algorithm. In section 4 we illustrate the performance of the algorithm by some simulation examples. Section 5 discusses the main advantages and drawbacks of the proposed technique, and suggests some possible modifications.

2. THE MODEL

Let $\{y_t\}$ be a p-dimensional zero-mean wide-sense stationary process. We assume that $\{y_t\}$ is related to some p-dimesnional white noise process $\{w_t\}$ by a rational pxp transfer matrix H(z), i.e.

$$y(z) = H(z)w(z), \tag{1}$$

where y(z) and w(z) are the formal z-transforms of $\{y_t\}$ and $\{w_t\}$. The transfer matrix H(z) is assumed to be stable and causal. The covariance matrix of w_t can be assumed, without loss of generality, to be the identify matrix (since this covariance matrix can always be absorbed in H(z).

The model (1) includes many common parametric models as special cases. For example, the ARMA model

$$y(z) = A_L^{-1}(z)B_L(z)w(z) \ , \tag{2}$$

is clearly of the form (1). In this case H(z) is written in the form of a left matrix fraction description (MFD). The AR plus noise model

$$x(z) = A_L^{-1}(z)u(z) \tag{3a}$$

$$y(z) = x(z) + v(z) \ , \tag{3b}$$

where $\{u_t\}$ and $\{v_t\}$ are uncorrelated white noise sequences, can also be transformed to the form (1). In some applications, the natural description of the process is in terms of a right MFD,

$$y(z) = B_R(z)A_R^{-1}(z)\ w(z) \ , \tag{4}$$

see e.g. [9] for such an application.

The main difficulty in using the models (2), (3) or (4) for spectral estimation lies in the fact that the denominators of these models are polynomial matrices. Therefore, the modes of the spectrum do not appear explicitly, but are "hidden" in the determinant of the corresponding matrix polynomial ($A_L(z)$ or $A_R(z)$). An alternative model, which makes it easier to display the spectral modes, is given by

$$H(z) = \frac{B(z^{-1})}{a(z^{-1})} \ , \tag{5}$$

where $a(z^{-1})$ is the least common multiple of the denominators of the entries of H(z), expressed in powers of z^{-1}. The matrix polynomial $B(z^{-1})$ is also expressed in powers of z^{-1}. In general, $a(z^{-1})$ and $B(z^{-1})$ of the same degree,

$$a(z^{-1}) = 1 + a_1 z^{-1} + \ldots + a_n z^{-n} \tag{6a}$$

$$B(z^{-1}) = B_0 + B_1 z^{-1} + \ldots + B_n z^{-n} \ . \tag{6b}$$

We note that while the model (5) is quite general, it is usually overparametrized. For example, let us compare the number of parameters in (5) to the number of parameters in the ARMA model (2). A p-dimensional ARMA(m,m) model has $(2m+1)p^2$ free parameters. The corresponding characteristic polynomial has degree $n=mp$, so that in (5) we have $mp+(mp+1)p^2$ parameters. For $p = 2$ we have $8m+4$ and $10m+4$ parameters, respectively. Thus,

for two-channel time series, the model (5) is only slightly overparametrized.

Let $S(z)$ be the spectral density matrix of the process $\{y_t\}$. This matrix is given by

$$S(z) = \frac{1}{a(z^{-1})a(z)} B(z^{-1})B^T(z) = \frac{1}{a(z^{-1})a(z)} N(z) , \tag{7}$$

where

$$N(z) = B(z^{-1})B^T(z) = N_{-n}z^n + \dots + N_1 z + N_0 + N_1 z^{-1} + \dots + N_n z^{-n} . \tag{8}$$

Next we write $S(z)$ in terms of the covariance sequence $\{R_i\}$,

$$R_i = E\{y_t y_{t-i}^T\} = R_{-i}^T , \quad -\infty < i < \infty . \tag{9}$$

Let

$$S_+(z^{-1}) = \frac{1}{2} R_0 + \sum_{i=1}^{\infty} R_i z^{-i} \tag{10}$$

be the causal part of the spectrum. Clearly,

$$S(z) = S_+(z^{-1}) + S_+^T(z) . \tag{11}$$

The causal part can be expressed as

$$S_+(z) = \frac{1}{a(z^{-1})} C(z^{-1}) , \tag{12}$$

where

$$C(z^{-1}) = C_0 + C_1 z^{-1} + \dots + C_n z^{-n} \; . \tag{13}$$

From (7),(11) and (12) we see that $N(z)$, $C(z^{-1})$ and $a(z^{-1})$ are related by

$$N(z) = C(z^{-1})a(z) + C^T(z)a(z^{-1}) \; . \tag{14}$$

As we will see in the next section, the parameters of the matrix $C(z^{-1})$ can be estimated by a relatively simple procedure.

3. THE ALGORITHM

The proposed algorithm is based on estimating the coefficients of the rational spectral density matrix from the sample covariances. These are computed from the measured data by

$$\hat{R}_i = K(T,i) \sum_{t=i+1}^{T} y_t y_{t-i}^T , \tag{15}$$

where $K(T,i)$ is either $1/T$ or $1/(T-i)$ (the biased or unbiased covariances). Each sample covariance $\hat{R}_i$ is a pxp matrix. We also define for each i a p^2-dimensional column vector $\hat{\rho}_i$ obtained by stacking the columns of $\hat{R}_i$, i.e.,

$$\hat{\rho}_i = [\hat{R}_i(1,1),\ldots,\hat{R}_i(p,1),\ldots,\hat{R}_i(1,p),\ldots,\hat{R}_i(p,p)]^T . \tag{16}$$

Note that any estimation algorithm based on sample covariances will not be efficient in the statistical sense, i.e., it will not achieve the Cramer-Rao lower bound, even asymptotically* [10]. However, by increasing the number of sample covariances used in the algorithm the loss of efficiency can be made quite small [11]. Furthermore, spectral estimation algorithms based on sample covariances are known to be more robust than algorithms of the maximum likelihood type (i.e., they are less sensitive to initial conditions, model inaccuracy, or the choice of the number of parameters).

The algorithm consists of three steps. In the first step, an initial estimate of the characteristic polynomial is obtained by a multichannel version of the modified Yule-Walker equations. This initial estimate has a

*Except in the special case of pure autoregressive processes.

high degree in general, compared to the degree of the true characteristic polynomial. Thus, at the second step, the degree of the initial estimate is reduced to yield the final estimate of the characteristic polynomial. This is done using modal decomposition of the causal part of the spectrum and an appropriate elimination process. Finally, in the third step, the numerator matrices are estimated by a least squares technique, using the estimates of the characteristic polynomial from the second step.

The three steps of the algorithm will now be described in greater detail.

3.1 INITIAL ESTIMATION OF THE CHARACTERISTIC POLYNOMIAL

The covariances of the process $\{R_i\}$ can be easily shown to satisfy the Yule-Walker type equations

$$\sum_{k=1}^{n} a_k R_{i-k} = -R_i \; ; \quad i > n+1. \tag{17}$$

Substituting the sample covariances for the true covariances in (17) we get the so-called modified Yule-Walker equations

$$\sum_{k=1}^{n} a_k \hat{R}_{i-k} = -\hat{R}_i \; ; \quad i > n+1 \; , \tag{18}$$

or equivalently,

$$\sum_{k=1}^{n} a_k \hat{\rho}_{i-k} = -\hat{\rho}_i \; ; \quad i > n+1. \tag{19}$$

It was demonstrated experimentally in [12], and proven mathematically in [13], that by taking an overdetermined set of equations of the form (19) and solving them in the least-squares sense, the statistical efficiency of the estimated

characteristic polynomial coefficients can be improved, compared to the case where only the minimal number of equations is used. Thus, in practice we solve the following set of equations in the least-squares sense:

$$G = \begin{bmatrix} \hat{\rho}_{n_2} & \cdots\cdots\cdots & \hat{\rho}_2 & \hat{\rho}_1 \\ \hat{\rho}_{n_2+1} & \cdots\cdots & \hat{\rho}_3 & \hat{\rho}_2 \\ \vdots & & \vdots & \vdots \\ \hat{\rho}_{n_1+n_2-1} & \cdots & \hat{\rho}_{n_1+1} & \hat{\rho}_{n_1} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n_2} \end{bmatrix} = - \begin{bmatrix} \hat{\rho}_{n_2+1} \\ \hat{\rho}_{n_2+2} \\ \vdots \\ \hat{\rho}_{n_1+n_2} \end{bmatrix} . \qquad (20)$$

The number of equations in (20) is $p^2 n_1$, and the number of unknowns is n_2 . It was shown in [8], [14] that, if n_2 is taken as the true degree of the process characteristic polynomial, the estimates $\{\hat{a}_1, \ldots, \hat{a}_n\}$ may be considerably biased in some cases. This can be intuitively explained as follows: Equation (20) has the form of a least-squares autoregressive fit of the "data" $\{\hat{\rho}_1, \hat{\rho}_2, \ldots\}$. It is well known that the estimates $\{\hat{a}_1, \ldots, \hat{a}_n\}$ are unbiased only when the error between the two sides of (20) is a white noise. However, for general rational models, the sequence $\{\hat{\rho}_1, \hat{\rho}_2, \ldots\}$ does not follow an exact autoregression, and the error will not be white. By taking a sufficiently large order n_2 in (20), we can approximately "whiten" the error sequence. Based on this intuitive argument, the use of Akaike's information criterion was advocated in [8] to determine the value of n_2 . Here we follow the same choice, but mention that other choices have been proposed, e.g. [15],[16].

The number of equations in (20) is usually selected by some ad-hoc

procedure. However, $p^2n_1 >> n_2$ is often necessary to guarantee a reasonable statistical efficiency, see e.g. [12],[13]. We have adopted a constant ratio between p^2n_1 and n_2 for convenience. Thus, equation (20) is solved for different values of n_2, where p^2n_1 is always taken to be fixed multiple of n_2. For each solution, the Akaike information criterion [17] is computed, and the final choice of n_2 is made by minimizing this criterion.

3.2 ORDER REDUCTION BY MODAL DECOMPOSITION

Let us denote the n_2-th order polynomial obtained from the modified Yule-Walker equations by $\tilde{a}(z^{-1})$. As explained before, the degree of $\tilde{a}(z^{-1})$ is usually much larger than the true degree of the characteristic polynomial. Furthermore, it was shown in [8] that a final estimate of the characteristic polynomial can be chosen to be a divisor of $\tilde{a}(z^{-1})$. This divisor is obtained by the following process of decomposition and elimination.

Let us factor $\tilde{a}(z^{-1})$ into its first- and second-order real factors:

$$\tilde{a}(z^{-1}) = \left(\prod_{i=1}^{n_r} d_i(z^{-1})\right)\left(\prod_{i=1}^{n_c} e_i z^{-1})\right) \quad , \tag{21}$$

where n_r is the number of real roots and n_c is the number of complex pairs of roots, so that $n_2 = n_r + 2n_c$. The polynomials $\{d_i(z^{-1})\}$ are of degree 1, and the polynomials $\{e_i(z^{-1})\}$ are of degree 2, i.e.,

$$d_i(z^{-1}) = 1 + d_{1,i}z^{-1} \quad , \tag{22a}$$

$$e_i(z^{-1}) = 1 + e_{1,i}z^{-1} + e_{2,i}z^{-2} \quad . \tag{22b}$$

Since $\tilde{a}(z^{-1})$ is not guaranteed to be stable, it is necessary to replace it by

a stable spectral factor of $\tilde{a}(z^{-1})\tilde{a}(z)$. This is done by reflecting the unstable roots of $\tilde{a}(z)$ inside the unit circle, as follows: whenever $|d_{i,1}| > 1$ we redefine $d_i(z)$ as

$$d_i(z^{-1}) = 1 + \frac{1}{d_{1,i}} z^{-1} . \tag{23a}$$

Similarly, whenever $|e_{2,i}| > 1$, we redefine $e_i(z)$ as

$$e_i(z^{-1}) = 1 + \frac{e_{1,i}}{e_{2,i}} z^{-1} + \frac{1}{e_{2,i}} z^{-2} . \tag{23b}$$

Assuming that all the roots of $\tilde{a}(z^{-1})$ are distinct, we can use (10) to make the following approximation,

$$\sum_{i=1}^{n_3} \hat{R}_i z^{-i} \approx \left\{ \sum_{j=1}^{n_r} \frac{D_{1,j} z^{-1}}{1 + d_{1,j} z^{-1}} + \sum_{j=1}^{n_c} \frac{E_{1,j} z^{-1} + E_{2,j} z^{-2}}{1 + e_{1,j} z^{-1} + e_{2,j} z^{-2}} \right\} * \Pi(z), \tag{24}$$

where $\Pi(z)$ is the z-transform of a rectangular window on the interval $[1, n_3]$, and * denotes a complex convolution. The number of covariances n_3 is chosen so that $n_3 >> n_2$.

The expansion (24) will be used to select modes that will appear in the final estimated characteristic polynomial, and to eliminate undesired modes. To do this we estimate $\{D_{1,j}\}$ and $\{E_{1,j}, E_{2,j}\}$ by performing the following least-squares fit in (24): let us denote

$$\frac{1}{d_j(z^{-1})} = \sum_{i=0}^{\infty} f_{i,j} z^{-i} , \tag{25a}$$

$$\frac{1}{e_j(z^{-1})} = \sum_{i=0}^{\infty} g_{i,j} z^{-i} . \tag{25b}$$

Then we can express (24) in the time domain as

$$[F\ G]\begin{bmatrix}\tilde{D}\\ \tilde{E}\end{bmatrix} = \begin{bmatrix}\hat{\rho}_1\\ \hat{\rho}_2^T\\ \vdots\\ \hat{\rho}_{n_3}^T\end{bmatrix}, \tag{26a}$$

where

$$F = \begin{bmatrix} f_{0,1} & \cdots\cdots & f_{0,n_r}\\ f_{1,1} & \cdots\cdots & f_{1,n_r}\\ \vdots & & \vdots\\ f_{n_3-1,1} & \cdots & f_{n_3-1,n_r}\end{bmatrix} \tag{26b}$$

$$G = \begin{bmatrix} g_{0,1} & 0 & \cdots\cdots & g_{0,n_c} & 0\\ g_{1,1} & g_{0,1} & & g_{1,n_c} & g_{0,n_c}\\ \vdots & \vdots & & \vdots & \vdots\\ g_{n_3-1,1} & g_{n_3-2,1} & \cdots & g_{n_3-1,n_c} & g_{n_3-2,n_c}\end{bmatrix} \tag{26c}$$

(do by hand)

$$\tilde{D} = \begin{bmatrix}\delta_{1,1}^T\\ \delta_{1,2}^T\\ \vdots\\ \delta_{1,n_r}^T\end{bmatrix};\quad \tilde{E} = \begin{bmatrix}\epsilon_{1,1}^T\\ \epsilon_{2,1}^T\\ \vdots\\ \epsilon_{2,n_c}^T\end{bmatrix}. \tag{26d}$$

The vectors $\delta_{1,j}$ are obtained by stacking the columns of $D_{1,j}$, the vectors $\epsilon_{1,j}$ by stacking the columns of $E_{1,j}$, and the vectors $\epsilon_{2,j}$ by stacking the

columns of $E_{2,j}$ (cf. equation (16)). Equation (26a) is now solved in the least-squares sense and the solutions are "unstacked" to yield $\{D_{1,j}\}$ and $\{E_{1,j}, E_{2,j}\}$.

The next step is to compute the energies of the individual modes in the various channels. It is not difficult to show that these energies are given by the following formulas.

$$\text{Energy of j-th real mode in channel } \#\ell = \frac{[(D_{1,j})_{\ell,\ell}]^2}{1 - d_{1,j}^2} , \tag{27a}$$

Energy of j-th complex mode in channel $\#\ell$ =

$$\frac{(1+e_{2,j})\{[(E_{1,j})_{\ell,\ell}]^2+[(E_{2,j})_{\ell,\ell}]^2\}-2e_{1,j}[(E_{1,j})_{\ell,\ell}][(E_{2,j})_{\ell,\ell}]}{(1-e_{2,j})(1+e_{2,j}+e_{1,j})(1+e_{2,j}-e_{1,j})}. \tag{27b}$$

Typically, the true modes (i.e.. those present in the actual spectrum) will tend to have relatively high energies, while spurious modes will have relatively low energies. We therefore arrange the $(n_r+n_c)p$ energies in order of decreasing magnitudes, and associate each energy with its "parent mode". The mode selection process can now be done, using either of the two following criteria:

(i) Energy threshold criterion.

In this case all modes whose energies are above a certain threshold are retained, and the other modes are discarded. It is convenient to measure all the energies in dB relative to the highest energy, and then a reasonable threshold would be, e.g., -50 dB.

(ii) Order criterion.

In this case a pre-selected number of modes (corresponding to the highest energies) is retained, and the rest are deleted. This is convenient in cases where the true order of $a(z^{-1})$ is known a priori.

Finally, all modes chosen to be retained are multiplied out to form the final estimated characteristic polynommial $\hat{a}(z^{-1})$. Clearly, $\hat{a}(z^{-1})$ is a divisor of $\tilde{a}(z^{-1})$. Also, the mode selection procedure described above guarantees that these modes capture most of the signal energy, in the sense of the approximation (24).

3.3 ESTIMATION OF THE NUMERATOR

Numerator estimation is based on the additive decomposition (11),(12). Similarly to (24), we can approximate $\hat{S}_+(z^{-1})$ by

$$\frac{1}{2}\hat{R}_0 + \sum_{i=1}^{n_4} \hat{R}_i z^{-i} \approx \{\frac{1}{\hat{a}(z^{-1})} \hat{C}(z^{-1})\} * \Pi(z) \quad , \tag{28}$$

where $\Pi(z)$ is now the z-transform of a rectangular window on the interval $[0, n_4]$. The number n_4 can be taken to be much smaller than n_3, because the order of $\hat{a}(z^{-1})$ is usually much smaller than that of $\tilde{a}(z^{-1})$. Let us denote,

$$\frac{1}{\hat{a}(z^{-1})} = \sum_{i=0}^{\infty} h_i z^{-i} \ . \tag{29}$$

Then we can rewrite (28) as

$$\begin{bmatrix} h_0 & & & 0 \\ h_1 & h_0 & & \\ \vdots & & \ddots & \\ h_{\hat{n}} & \cdots & \cdots & h_0 \\ \vdots & & & \vdots \\ h_{n_4} & \cdots & \cdots & h_{n_4-\hat{n}} \end{bmatrix} \begin{bmatrix} \hat{\gamma}_0^T \\ \hat{\gamma}_1^T \\ \vdots \\ \hat{\gamma}_{\hat{n}}^T \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\hat{\rho}_0^T \\ \hat{\rho}_1^T \\ \vdots \\ \hat{\rho}_{n_4}^T \end{bmatrix}, \qquad (30)$$

where the vectors $\hat{\gamma}_i$ are obtained by stacking the columns of $\hat{C}_i$. Equation (30) is solved in the least-squres sense, and the solutions are "unstacked" to form $\{\hat{C}_i\}$. Finally, the numerator of the spectral density matrix is computed by

$$\hat{N}(z) = \hat{C}(z^{-1})\hat{a}(z) + \hat{C}^T(z)\hat{a}(z^{-1}). \qquad (31)$$

The estimated numerator $\hat{N}(z)$ and denominator $\hat{a}(z^{-1})$ can be inserted into equation (7) to provide the desired spectral estimate.

4. SIMULATION EXAMPLES

The algorithm described in the previous section was programmed and tested for various types of two-channel processes. Here we illustrate the performance of the algorithm by three examples. In all examples the number of data points was T = 1024, and n1 = 20, n2 = 16, n3 = 16, n4 = 8.

Example #1:

Here we generated the data by a right MFD $B_R(z)A_R^{-1}(z)$, where

$$A_R(z) = \begin{bmatrix} 1-1.4z^{-1} + 0.95z^{-2} & 0 \\ 0.3z^{-1} + 0.6z^{-2} & 1+0.6z^{-1}+0.95z^{-2} \end{bmatrix} , \tag{32a}$$

$$B_R(z) = \begin{bmatrix} 1 + 0.3z^{-1} + 0.4z^{-2} & 0.8z^{-1} + 0.3z^{-2} \\ 0 & 1 + 0.6z^{-2} \end{bmatrix} . \tag{32b}$$

Figure 1a shows the autospectra $S_{11}(\omega)$ and $S_{22}(\omega)$, and the co-spectrum $S_{21}(\omega)$ (magnitude and phase) of this model. Figure 1b shows the corresponding estimates obtained by the algorithm. As we see, the estimates match fairly closely the true spectra, except at frequencies where the energy density is very low. This is not surprising, since any estimation based on least-squares fit would give little weight to low energy regions.

Example #2:

Here we generated data according to the model

$$y_1(t) = \sqrt{2} \sin 2\pi f_1 t + \frac{1}{2} \sqrt{2} \sin 2\pi f_2 (t-D) + n_1(t) , \tag{33a}$$

$$y_2(t) = \sqrt{2} \sin 2\pi f_2 t + n_2(t) , \tag{33b}$$

where $n_1(t)$, $n_2(t)$ are uncorrelated zero-mean white-noise sequences, with unit variance. The frequencies f_1, f_2 were 0.12 Hz and 0.18 Hz respectively. The time-delay D is 2 seconds.

The estimated autospectra and co-spectrum are shown in Figure 2. Note that both sinusoids are well represented in $S_{11}(\omega)$. Also note the slight "leakage" of the first sinusoid into the second channel. In the next section we comment on how such "leakage" can be avoided.

Example #3:

Here we generated the data by the left MFD $A_L^{-1}(z)B_L(z)$, where

$$A_L(z) = \begin{bmatrix} 1 - z^{-1} + 0.8z^{-2} & z^{-2} \\ -0.02z^{-1} + 0.01z^{-2} & 1-1.2z^{-1} + z^{-2} \end{bmatrix} , \tag{34a}$$

$$B_L(z) = \begin{bmatrix} 1 + 0.2z^{-1} & -0.3z^{-1} - 0.1z^{-1} \\ 0.4 + 0.1z^{-1} & 1 + 0.3z^{-1} \end{bmatrix} . \tag{34b}$$

The characterisitc polynomial in this example has two narrowband modes with relatively closed frequencies, as shown in the ture spectra in Figure 3a. Also, both channels have relatively low energies at the high frequency band. As we see in Figure 3b, the estimates are fairly accurate in the low frequency band, but quite inaccurate in the high frequency band. Evidently, the algorithm has problems in adequately representing the frequencies where the energy density is low.

Numerous other tests not shown here indicated a similar behavior: good accuracy at high energy regions, poor accuracy at low energy regions, and some "leakage" of energy from one channel to the other.

5. DISCUSSION

We presented a spectral estimation algorithm for multichannel stationary time series with rational spectra. The proposed algorithm is non-iterative and requires mostly the solution of linear sets of equations, except for the factorization of the estimated characteristic polynomial. The algorithm is fairly robust in the sense that

(i) No initial conditions are necessary;

(ii) Various types of rational models can be handled by the algorithm;

(iii) The model order need not be known a priori, but is estimated by the algorithm.

The main disadvantages of the algorithm appear to be as follows:

(i) The algorithm is not efficient in the statistical sense;

(ii) The accuracy of the estimates in frequencies of low energy densities is poor;

(iii) Some inter-channel "leakage" is apparent;

(iv) Positive definiteness of $S(z)$ on the unit circle is not guaranteed.

Point (i) is inherent to any algorithm based on the sample covariances. Point (ii) is also typical to many algorithms based on sample covariances, especially those which are based on some least-squares fit. Point (iii) can be largely solved by the following modification of the algorithm: instead of performing mode selection using the diagonal elements of $S_+(z)$, we can perform individual mode selections for the p^2 elements of this matrix to obtain p^2 different denominators, instead of one common denominator. The

various thresholds used in this selection procedure can be adjusted so as to eliminate any undesired leakage. The improved version of the algorithm is currently under investigation, and results will be reported later.

As to point (iv) the only way to guarantee positive definiteness of the spectrum appears to be by direct estimation of the spectral factor $B(z^{-1})$. This leads, however, to a nonlinear problem, which requires some iterative techniques for its solution - see e.g. [18], [19]. Procedures for estimating the spectral factor $B(z^{-1})$ appear to be inherently more complex than techniques for estimating $B(z^{-1})B^T(z)$.

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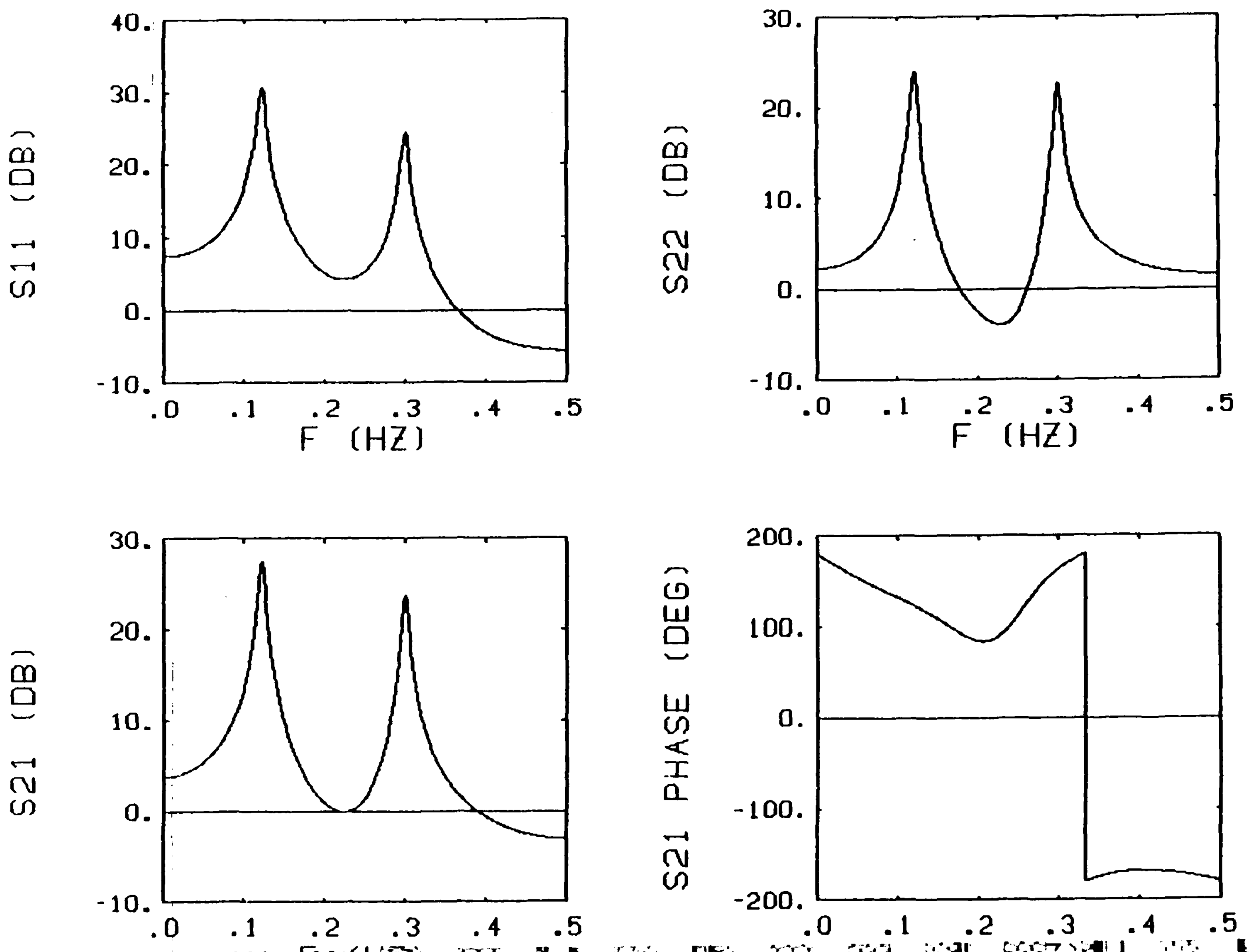
Figure 1a   Example #1: right MFD - true spectrum

Figure 1b: Example #1: right MFD - estimated spectrum

Figure 2:   Example #2: sinusoids in noise - estimated spectrum

Figure 3a: Example #3: left MFD - true spectrum

Figure 3b: Example #3: left MFD - estimated spectrum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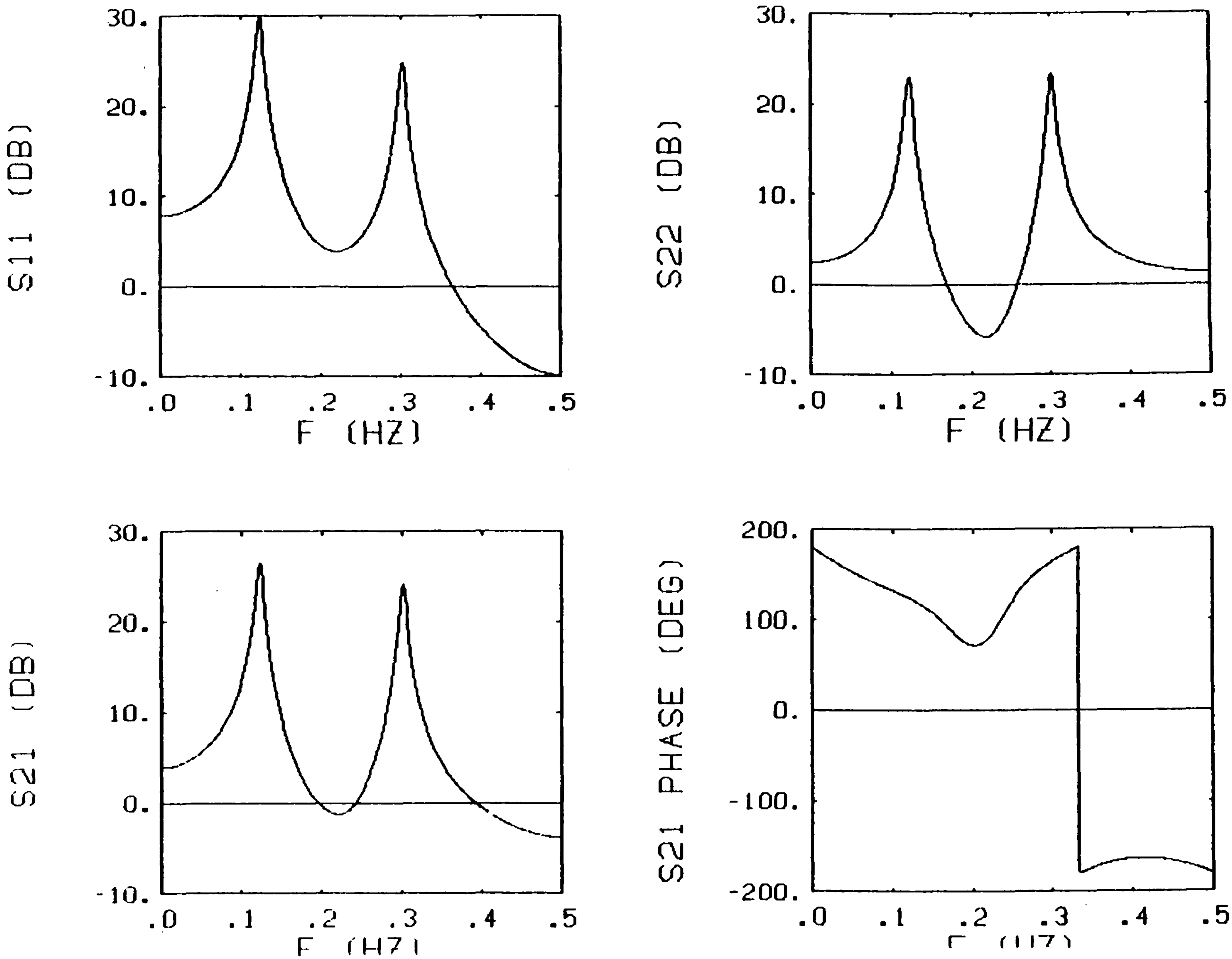

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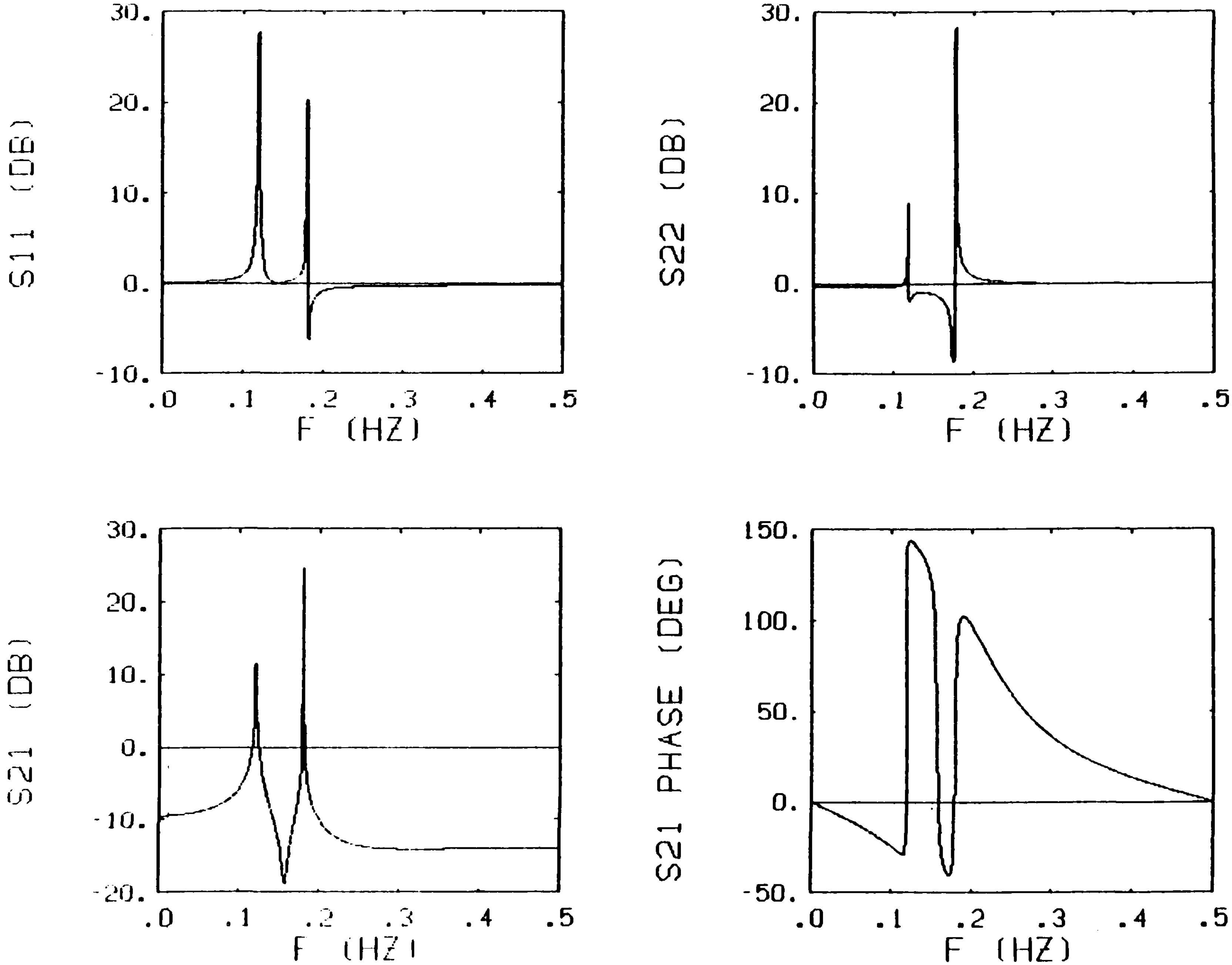
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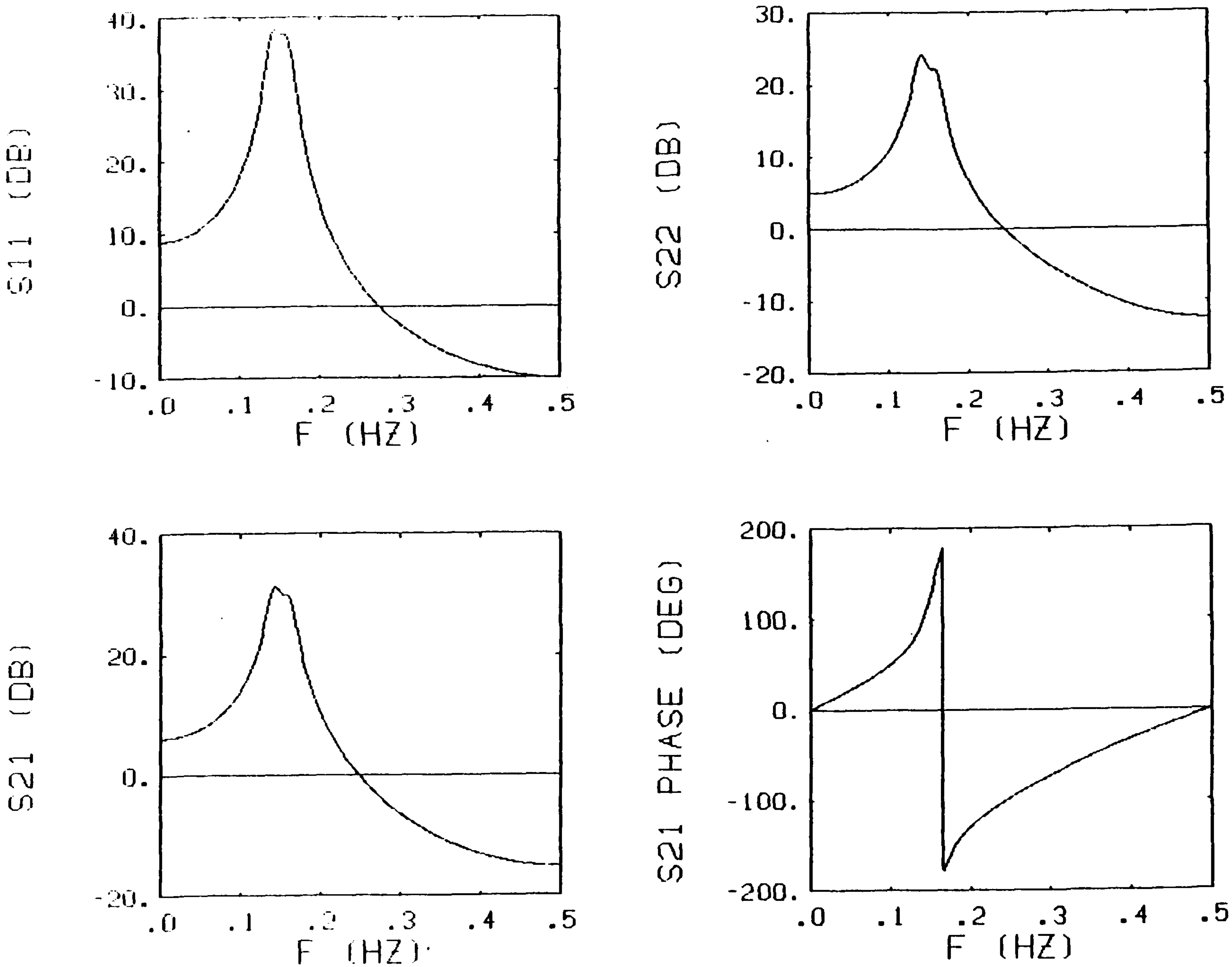

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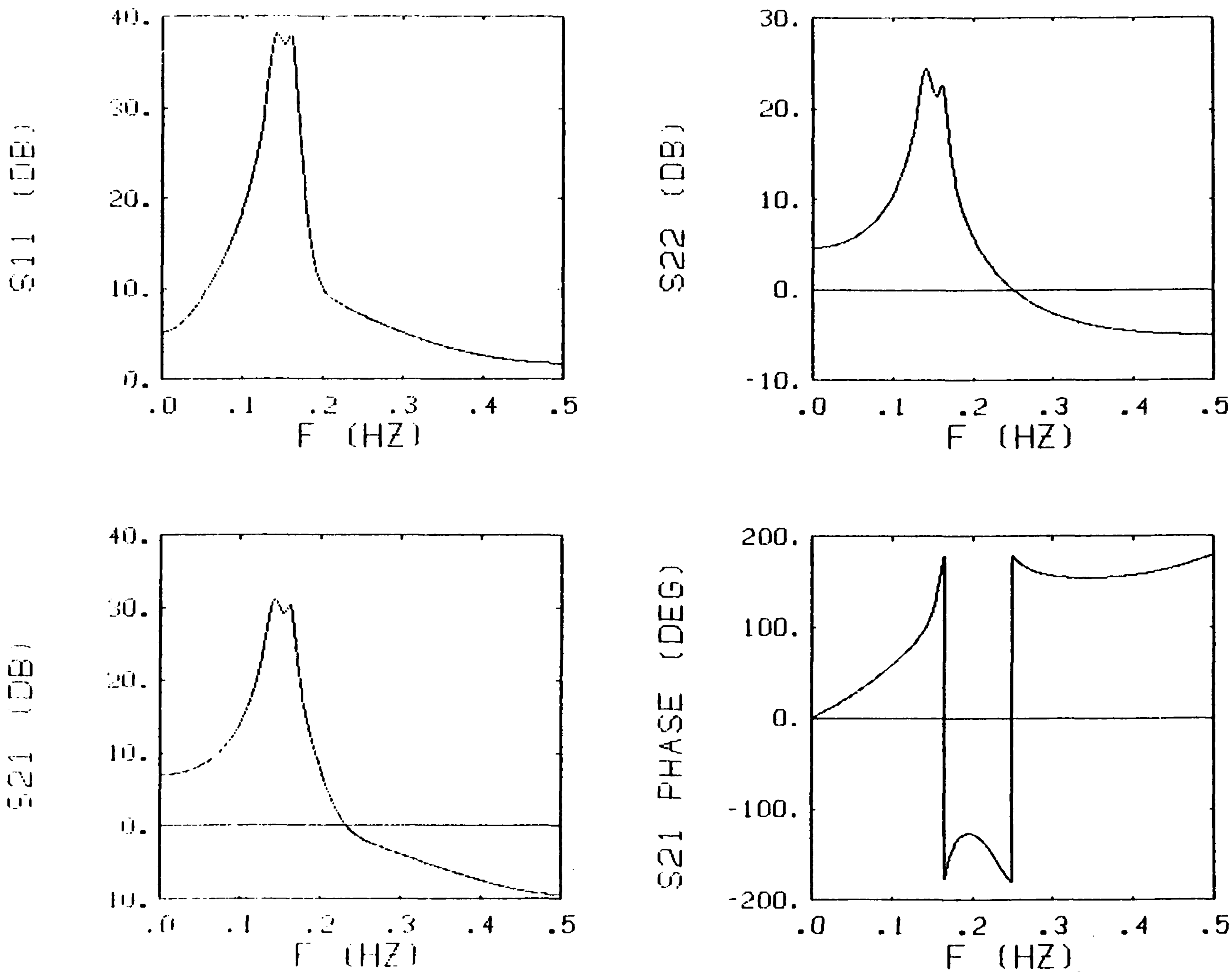
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APPENDIX F

OPTIMAL INSTRUMENTAL VARIABLE MULTISTEP ALGRITHMS FOR ESTIMATION OF THE AR PARAMETERS OF AN ARMA PROCESS

Revised 10/85

OPTIMAL INSTRUMENTAL VARIABLE MULTISTEP ALGORITHMS FOR ESTIMATION OF THE AR PARAMETERS OF AN ARMA PROCESS

P. Stoica, B. Friedlander and T. Söderström

ABSTRACT

Multistep implementations are derived for the optimal instrumental variable (OIV) estimators introduced in [1]. The proposed algorithms provide asymptotically efficient estimates of the AR parameters of an ARMA process. The computational complexity of these algorithms is modest compared to the (exact) maximum likelihood estimator. The performance of the OIV algorithms is illustrated by some numerical examples.

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1. INTRODUCTION

The need for estimating the parameters of an autoregressive moving-average (ARMA) process arises in many applications in the areas of signal processing, spectral analysis, estimation and system identification. A computationally attractive estimation procedure, which has received considerable attention in the literature, is based on a two-step approach: first the autoregressive (AR) parameters are estimated using the modified Yule-Walker (MYW) equations; then the moving average (MA) parameters are estimated by one of several available techniques [2]-[7].

In this paper we consider only the first step of estimating the autoregressive parameters. In many engineering applications the second estimation step is not needed. The prime example is the estimation of autoregressive signals corrupted by white measurement noise. In this case all the information about the spectral shape of the signal lies in the AR parameters of the signal-plus-noise ARMA process.

In a companion paper [1] we presented a number of results related to the asymptotic accuracy of a fairly general class of instrumental variable (IV) estimators, which includes the MYW estimator as a special case. In particular, it was shown that estimation accuracy increases monotonically with the number of MYW equations for an optimal choice of the weighting matrix used in the least squares solution of these equations. Furthermore, the asymptotic error covariance of the optimal IV method equals that of the prediction error method. In other words, the optimal IV method is asymptotically (as the number of data points and the number of MYW equations tend to infinity) efficient. An alternative form of the optimal IV method involving pre-filtering of the data used in the instrument vector while using a minimal number of MYW equations was also discussed.

The main difficulty associated with the optimal IV method is that the optimal weighting matrix, and the optimal pre-filter, depend on the second-order statistics of the data, which are not known a-priori. The objective of this paper is to propose several multistep algorithms for overcoming this difficulty. As we will show, these algorithms provides asymptotically

efficient estimates of the AR parameters.

The structure of the paper is as follows. In section 2 we briefly review the results on optimal instrumental variable (OIV) estimation derived in [1]. Three approximate implementations of the OIV are presented and analyzed in section 3: one based on an optimal weighting matrix and two based on an optimal pre-filtering operation. The implementation of these forms of the OIV estimator by means of a multi-step procedure is discussed in section 4. The performance of the proposed estimation techniques is studied in section 5 by means of some numerical examples.

The work presented here and in [1] provides an extension of IV methods which are usually applied to system identification, to problems of time series analysis. For an overview of IV methods and their applications see [11]-[12].

2. THE OPTIMAL IV ESTIMATES

Consider the following ARMA process of order (na, nc)

$$A(q^{-1})y(t) = C(q^{-1})e(t) \,, \qquad (1)$$

where

$e(t)$ = white noise process with zero mean and variance λ^2 ,

$A(q^{-1}) = 1 + a_1 q^{-1} + \ldots + a_{na} q^{-na}$,

$C(q^{-1}) = 1 + c_1 q^{-1} + \ldots + c_{nc} q^{-nc}$,

q^{-1} = unit delay operator $(q^{-1}y(t)=y(t-1))$.

The following assumptions are made:

A1: $A(z) = 0 = |z| > 1$; $C(z) = 0 = |z| > 1$.

In other words, the ARMA representation (1) is stable and invertible.

This is not a restrictive assumption (cf. the spectral factorization theorem, e.g., [16]).

A2: $a_{na} \neq 0$, $c_{nc} \neq 0$, and $\{A(z), C(z)\}$ are coprime polynomials.

In other words, (na, nc) are the minimal orders of the ARMA model (1).

Next we introduce the notation:

$$\begin{aligned} \phi(t) &= [-y(t-1),\ldots,-y(t-na)]^T, \\ \theta &= [a_1,\ldots,a_{na}]^T, \\ v(t) &= C(q^{-1})e(t). \end{aligned} \tag{2}$$

Then equation (1) can be rewritten as

$$y(t) = \phi^T(t)\theta + v(t) \tag{3}$$

The unknown parameter vector θ will be estimated by minimizing a quadratic cost function involving the data vector $\phi(t)$ and an IV vector $z_m(t)$:

$$\hat{\theta} = \arg\min_{\theta} \left\| \left[\sum_{t=1}^{N} z_m(t)\phi^T(t)\right]\theta - \left[\sum_{t=1}^{N} z_m(t)y(t)\right] \right\|_Q, \tag{4}$$

where

N = number of data points,

$\|x\|_Q^2 \triangleq x^T Q x$, Q = a positive definite matrix ,

and

$$z_m(t) = G(q^{-1}) \begin{bmatrix} y(t-nc-1) \\ \vdots \\ y(t-nc-m) \end{bmatrix}, \qquad m \geq na , \tag{5}$$

where $G(q^{-1})$ is a rational filter. We assume that

A3: $G(q^{-1})$ is stable and invertible, and $G(0) = 1$.

It is straightforward to show that the IV estimate in equation (4) can be obtained by a least-squares solution of the following system of linear equations:

$$Q^{1/2}\Big[\sum_{t=1}^{N} z_m(t)\phi^T(t)\Big]\hat{\theta} = Q^{1/2}\Big[\sum_{t=1}^{N} z_m(t)y(t)\Big] , \tag{6}$$

where $Q^{1/2}$ is a matrix square-root of the weighting matrix Q (i.e., $Q = Q^{T/2}\,Q^{1/2}$) . The class of estimates defined by (6) includes the various MYW estimation techniques discussed in the literature as special case; see [1] for details.

It was shown in [1], that under conditions A1-A3, the covariance P_m of normalized estimation error $(\sqrt{N}/\lambda)(\hat{\theta}-\theta)$ obeys the inequality

$$P_m \geq \tilde{P}_m \triangleq [R_m^T\, S_m^{-1}\, R_m]^{-1} , \tag{7}$$

where

$$R_m = E\{z_m(t)\phi^T(t)\} , \quad m \times na , \tag{8}$$

$$S_m = E\{[C(q^{-1})z_m(t)][C(q^{-1})z_m(t)]^T\},\ m \times m. \tag{9}$$

Equality in (7) can be shown to hold for the "optimal" weighting matrix

where
$$Q = S_m^{-1} \tag{10}$$

Furthermore, it was shown that

$$\tilde{P}_m \geq \tilde{P}_{m+1} , \quad \text{for all} \quad m \geq na . \tag{11}$$

The monotonically non-increasing sequence $\tilde{P}_m$ converges to a limit denoted by

$\tilde{P}_\infty$. This limit was shown to equal the (normalized) error covariance matrix P_{PEM} associated with the Prediction Error Method (PEM). See [15] for a discussion of the PEM and its properties. Here we note only that PEM is an efficient estimator, i.e., P_{PEM} equals the Cramér-Rao lower bound. Thus, the IV estimator in (6) is asymptotically efficient if we set $Q = S_m^{-1}$ and let $m \to \infty$. The asymptotic error covariance matrix $\tilde{P}_\infty$ does not depend on $G(q^{-1})$ and we will usually choose $G(q^{-1}) = 1$. See [1] for proofs of the statements above.

It was shown, however, that the rate of convergence of $\tilde{P}_m$ is affected by the choice of $G(q^{-1})$, see [1]. If we set $G(q^{-1})=1/C^2(q^{-1})$, then we get the fastest possible "convergence rate": $\tilde{P}_m = \tilde{P}_\infty$ for $m > na$. Note that for m=na the matrix Q in (6) does not effect the solution and can be set to Q=I.

Another interesting choice for $G(q^{-1})$ is $G(q^{-1}) = A(q^{-1})/C^2(q^{-1})$. In this case $S_m = \lambda^2 I_m$. Thus, the optimal weighting matrix is Q = I (the scaling factor $1/\lambda^2$ does not matter).

To summarize, we have (at least) three ways of generating optimal IV estimates using equation (6):

$$\text{OIV-1:}\quad Q = S_m^{-1} \,,\; G(q^{-1}) = 1 \,,\quad m \to \infty \qquad (12)$$

$$\text{OIV-2:}\quad Q = I,\; G(q^{-1}) = 1/C^2(q^{-1}) \,,\quad m = na \qquad (13)$$

$$\text{OIV-3:}\quad Q = I,\; G(q^{-1})=A(q^{-1})/C^2(q^{-1}),\; m \to \infty \qquad (14)$$

The problem is that both of these methods depend on knowledge of unknown quantities. This is the usual dilemma in accuracy optimization. Our aim here is to show how to overcome this difficulty for the case under consideration.

We will start by showing that replacing S_m (in OIV-1), $C(q^{-1})$ (in OIV-2) and $C(q^{-1}),A(q^{-1})$ (in OIV-3) by their consistent estimates, will not affect asymptotic estimation accuracy. Then we show how to obtain such consistent estimates of S_m and $C(q^{-1})$. The proposed estimation procedures are therefore based on estimating S_m or $C(q^{-1})$ and using these estimates in (6)

instead of the true S_m or $C(q^{-1})$. As we will see, the implementation of OIV-1 does not require explicit computation of $C(q^{-1})$. This may be advantageous in applications where only the AR parameters need to be estimated. A more detailed discussion of the proposed algorithms will be given in the following sections.

3. ANALYSIS OF THE OPTIMAL IV MULTISTEP ESTIMATORS

In this sections we analyze the asymptotic properties of OIV-1 and OIV-2 by techniques similar to those used in [1],[13].

3.1 APPROXIMATE OIV-1

Let $\hat{S}_m$ denote a consistent estimate of S_m . Let $\hat{\theta}_1$ denote the OIV-1 estimate (6),(12) for a given in (possibly m = m(N), where m(N) increases without bound as $N \to \infty$) and let $\hat{\hat{\theta}}_1$ be the approximate OIV-1 estimate with S_m replaced by $\hat{S}_m$. Then we can state the following theorem.

Theorem 3.1

Let assumptions A1-A3 be true, and assume also that $\hat{S}_m - S_m = O(1/\sqrt{N})^{\dagger}$ and that $[m(N)]^8/N$ as $N \to \infty$. Then $\hat{\theta}_1$ and $\hat{\hat{\theta}}_1$ are asymptotically equivalent. We will say that two consistent estimates of θ , say $\hat{\theta}_1$ and $\hat{\hat{\theta}}_1$, are asymptotically equivalent if

$$\hat{\theta}_1 - \hat{\hat{\theta}}_1 = O(1/N^{\beta}) \text{ for } \beta > 0.5$$

†We will use throughout the paper the notation $O(\varepsilon)$ to denote a random variable with standard deviation $K\varepsilon$, where ε is small and where K is a (fintie) constant independent of ε .

Proof:

From the assumption above it follows that for sufficiently large N we have

$$\hat{S}_m^{-1}=(S_m+ O(1/\sqrt{N}))^{-1}=(I + O(m/\sqrt{N}))^{-1}S_m^{-1}=S_m^{-1}+O(m^2/\sqrt{N}) \tag{15}$$

Note that the OIV-1 solution of (6) can be written explicitly as

$$\hat{\theta}_1 = [R_N^T S_m^{-1} R_N]^{-1}[R_N^T S_m^{-1}\{\frac{1}{N}\sum_{t=1}^{N} z_m(t)y(t)\}] \tag{16}$$

where

$$R_N \triangleq \frac{1}{N}\sum_{t=1}^{N} z_m(t)\phi^T(t) . \tag{17}$$

It is straightforward to show that

$$\hat{\theta}_1-\theta=[R_N^T S_m^{-1} R_N]^{-1}[R_N^T S_m^{-1}\{\frac{1}{N}\sum_{t=1}^{N} z_m(t)v(t)\}]=O(m/\sqrt{N}). \tag{18}$$

Similarly,

$$\hat{\hat{\theta}}_1-\theta=[R_N^T \hat{S}_m^{-1}R_N]^{-1}[R_N^T \hat{S}_m^{-1}\{\frac{1}{N}\sum_{t=1}^{N} z_m(t)v(t)\}]. \tag{19}$$

From (15), (18), (19) it follows that

$$\hat{\hat{\theta}}_1-\theta=\{R_N^T[S_m^{-1}+O(m^2/\sqrt{N})]R_N\}^{-1}\{R_N^T[S_m^{-1}+O(m^2/\sqrt{N})]$$

$$\frac{1}{N}\sum_{t=1}^{N} z_m(t)v(t)\} =$$

$$=\{R_N^T S_m^{-1} R_N+O(m^4/\sqrt{N})\}^{-1}\{R_m^T S_m^{-1}\frac{1}{N}\sum_{t=1}^{N} z_m(t)v(t)+O(m^4/N)\}$$

$$= (\hat{\theta}_1 - \theta) + O(m^5/N) \tag{20}$$

Since $m^5/N = (m/\sqrt{N}) \cdot (m^4/\sqrt{N})$, and since $m^4/\sqrt{N} \to 0$ by assumption, it follows that the second term in (20) goes to zero as $N \to \infty$, faster than the term $\hat{\theta}_1 - \theta$ (which is $O(m/\sqrt{N})$, cf. (18)).

The convergence of $\tilde{P}_m$ to $\tilde{P}_\infty$ may be slow, especially if C(z) has zeros close to the unit circle, see [1]. For the idealized OIV-1 estimate $\hat{\theta}_1$ we may then need to consider a large m in order to obtain good accuracy. For the practical estimate $\hat{\hat{\theta}}_1$ the situation is, however, different. If m is too large with respect to N then $\hat{\hat{\theta}}_1$ and $\hat{\theta}_1$ may not have the same distribution and thus $\hat{\hat{\theta}}_1$ may not be (asymptotically) optimal. Theorem 3.1 gives an upper bound on m ($m(N)=N^{1/8-\delta}$, $\delta > 0$) guaranteeing that $\hat{\theta}_1$ and $\hat{\hat{\theta}}_1$ are asymptotically equivalent However, no attempt has been made to give a tight bound. In fact this seems quite difficult since a tight bound would be problem dependent. In section 5 we discuss further this point and illustrate it by means of some simulations. It is shown there that the bound of theorem 3.1 is quite conservative. That is to say, $\hat{\theta}_1$ and $\hat{\hat{\theta}}_1$ may behave similarly for $m > N^{1/8}$. As explained earlier, one needs to consider large values of m when the convergence of $\tilde{P}_m$ to $\tilde{P}_\infty$ is slow.

3.2 APPROXIMATE OIV-2

Let $\hat{C}(q^{-1})$ denote a consistent estimate of $C(q^{-1})$. Let $\hat{\hat{\theta}}_2$ be the approximate estimate obtained by replacing $C(q^{-1})$ by $\hat{C}(q^{-1})$ in the OIV-2 estimate $\hat{\theta}_2$ (6),(13). Then we can state the following theorem.

<u>Theorem 3.2</u>

Let assumptions A1-A3 hold true and assume that

$\hat{c}_i - c_i = O(1/\sqrt{N})$, $i=1,\dots,nc$. Then $\hat{\theta}_2$ and $\hat{\hat{\theta}}_2$ are asymptocially equivalent.

<u>Proof:</u>

The OIV-2 solution of (6) can be written explicitly as

$$\hat{\theta}_2 = R_N^{-1}\{\frac{1}{N}\sum_{t=1}^{N} \frac{1}{C^2(q^{-1})} \phi(t-nc)\ y(t)\} \tag{21}$$

where

$$R_N = \frac{1}{N}\sum_{t=1}^{N} \frac{1}{C^2(q^{-1})} \phi(t-nc)\ \phi^T(t) \tag{22}$$

It is straightforward to show that

$$\hat{\theta}_2-\theta = R_N^{-1}\{\frac{1}{N}\sum_{t=1}^{N} \frac{1}{C^2(q^{-1})} \phi(t-nc)\ v(t)\} \tag{23}$$

and with

$$\hat{R}_N = \frac{1}{N}\sum_{t=1}^{N} \frac{1}{C^2(q^{-1})} \phi(t-nc)\ \phi^T(t). \tag{24}$$

that

$$\hat{\hat{\theta}}_2-\theta = [\hat{R}_N^{-1}\ \{\frac{1}{N}\sum_{t=1}^{N} \frac{1}{\hat{C}^2(q^{-1})} \phi(t-nc)\ v(t)\} =$$

$$R_N^{-1}\{\frac{1}{N}\sum_{t=1}^{N} \frac{1}{C^2(q^{-1})} \phi(t-nc)\ v(t)$$

$$-(\frac{2}{N}\sum_{t=1}^{N} \frac{1}{C^3(q^{-1})}[\phi(t-nc-1),\ldots,$$

$$\ldots\phi(t-2nc)]v(t))(\hat{C}-C)\}+O(1/N)=(\hat{\theta}_2-\theta)+O(1/N) \tag{25}$$

where

$$C = [c_1,\ldots,c_{nc}]^T,\ \hat{C} = [\hat{c}_1,\ldots,\hat{c}_{nc}]^T$$

Since $\hat{\theta}_2-\theta = O(1/\sqrt{N})$ it follows from (25) that $\hat{\theta}_2$ and $\hat{\hat{\theta}}_2$ are asymptotically equivalent. Note that here the choice of the number of MYW equations is not an issue, since optimality is achieved for $m = na$. However, the implementation of the OIV-2 estimator requires estimation of the

$C(q^{-1})$ polynomial, while this can be avoided when implementing OIV-1. Estimating $C(q^{-1})$ is not an easy task and one often wants to avoid it, if possible. The relative advantages and disadvantages of the two estimators are discussed further in the following sections.

3.3 APPROXIMATE OIV-3

Let $\hat{C}(q^{-1})$ and $\hat{A}(q^{-1})$ denote consistent estimates of $C(q^{-1})$ and $A(q^{-1})$ respectively. Let $\hat{\theta}_3$ be the approximate estimate obtain by replacing $C(q^{-1})$ and $A(q^{-1})$ by $\hat{C}(q^{-1})$ and $\hat{A}(q^{-1})$ in the OIV-3 estimate $\hat{\theta}_3$ (6), (14). Then we can state the following theorem.

<u>Theorem 3.3</u>

Let assumptions A1-A3 hold true and assume that $\hat{c}_i - c_i = O(1/\sqrt{n})$, $i=1,\dots,nc$ and $\hat{a}_1 - a_i = O(1/\sqrt{n})$, $i=1,\dots,na$. Then $\hat{\theta}_3$ and $\hat{\hat{\theta}}_3$ are asymptotically equivalent.

<u>Poof:</u>

Let

$$\hat{R}_N = \frac{1}{N}\sum_{t=1}^{N} \hat{z}_m(t)\phi^T(t) \tag{26}$$

where

$$\hat{z}_m(t) = \frac{\hat{A}(q^{-1})}{\hat{C}^2(q^{-1})}\begin{bmatrix} y(t-nc-1) \\ \vdots \\ y(t-nc-m)\end{bmatrix} \tag{27}$$

then

$$\hat{R}_N = R_N + O(1/\sqrt{N} \tag{28}$$

$$R_N = E\{z_m(t)\phi^T(t)\}$$

where $z_m(t)$ is as defined in (5) with $G(q^{-1}) = A(q^{-1})/C^2(q^{-1})$.

Thus,

$$\hat{\hat{\theta}}_3 \triangleq (\hat{R}_N^T \hat{R}_N)^{-1} \hat{R}_N^T \frac{1}{N} \sum_{t=1}^{N} \hat{z}_m(t)v(t) =$$

$$= \theta + (\hat{R}_N^T \hat{R}_N)^{-1} \hat{R}_N^T \frac{1}{N} \sum_{t=1}^{N} \hat{z}_m(t)v(t) \tag{29}$$

Now

$$(\hat{R}_N^T \hat{R}_N)^{-1} = (R_N^T R_N + O(m/\sqrt{N}))^{-1} = (R_N^T R_N)^{-1} + O(m/\sqrt{N}) \tag{30}$$

and

$$\frac{1}{N} \sum_{t=1}^{N} \hat{z}_m(t)v(t) = \frac{1}{N} \sum_{t=1}^{N} z_m(t)v(t) + O(1/N) \tag{31}$$

Thus

$$\hat{\hat{\theta}}_3 - \theta = [(R_N^T R_N)^{-1} + O(m/\sqrt{N})][R_N^T + O(1/\sqrt{N})]$$

$$[\frac{1}{N} \sum_{t=1}^{N} z_m(t)v(t) + O(1/N)] =$$

$$[(R_N^T R_N)^{-1} + O(m/\sqrt{N})][\underbrace{R_N^T \frac{1}{N} \sum_{t=1}^{N} z_m(t)v(t) + O(m/\sqrt{N})}_{O(m/\sqrt{N})}]$$

$$= (\hat{\theta}_3 - \theta) + O(m^2/N) \tag{32}$$

or

$$\hat{\hat{\theta}}_3 - \hat{\theta}_3 = O(m^2/N) \tag{33}$$

Since $(\hat{\theta}_3 - \theta) = O(m/\sqrt{N})$ (see (32)), we conclude that for $\hat{\hat{\theta}}_3$ and $\hat{\theta}_3$ to be asymptotically equivalent it is sufficient that

$$m/\sqrt{N} \to 0 \quad \text{as} \quad m,N \to \infty \tag{34}$$

In that case $m^2/N = (m/\sqrt{N})^2$ goes to zero faster than $m/\sqrt{N}$.

The requirement in (34) that $m/\sqrt{N} \to 0$ is not restrictive, since the fact that $\hat{\theta}_3 - \theta = O(m/\sqrt{N})$ will not be true if (34) does not hold.

The behavior of $\tilde{P}_m$ (for OIV-3) as m increases is quite different from its behavior in the case of OIV-1. By specializing the results in [1] to the case of OIV-3, it can be readily shown that $\tilde{P}_m^{-1}$ obeys the discrete-time Lyapunov equation

$$\tilde{P}_{m+1}^{-1} - A\,\tilde{P}_m^{-1}A^T = bb^T$$

where

$$A = \begin{bmatrix} -a_1, \ldots, -a_{na} \\ 1 \quad\quad\quad\quad \\ \quad \ddots \quad\quad 0 \\ 0 \quad\quad \ddots \quad 1 \end{bmatrix},$$

$$b = \frac{1}{\lambda} E\{\phi(t) \cdot \frac{1}{C(q^{-1})} e(t-nc-1)\}$$

From the equations above we conclude that the convergence rate of $\tilde{P}_m$ depends on the zeroes of $A(q^{-1})$, but not on the zeroes of $C(q^{-1})$. The reverse is true for OIV-1; see [1] for details. More specifically, $|\tilde{P}_\infty - \tilde{P}_m| \sim |\lambda_{MAX}|^{2m}$, where λ_{MAX} is the zero of $A(q^{-1})$ with the largest modulus. Thus, when $A(q^{-1})$ has roots closer to the unit circle than the zeroes of $C(q^{-1})$, we expect $\tilde{P}_m$ to converge faster for OIV-1 than for OIV-3 (and vice-versa when the zeroes of $C(q^{-1})$ are closer to the unit circle than the zeroes of $A(q^{-1})$) .

4. IMPLEMENTATION OF THE OPTIMAL IV MULTISTEP ESTIMATORS

4.1 THE OIV-1 ALGORITHM

Let us denote by $r_v(\tau)$ and $R_z(\tau)$ the covariances of $v(t)$ and $z_m(t)$, respectively:

$$r_v(\tau) = E\{v(t)\ v(t-\tau)\} \tag{35}$$

$$R_z(\tau) = E\{z_m(t)\ z_m^T\ (t-\tau)\} \tag{36}$$

Next note that,

$$S_M = E\{C(q^{-1})z_m(t)C(q^{-1})z_m^T(t)\} = E\{\sum_{i=0}^{nc}\ \sum_{j=0}^{nc}\ c_i c_j z_m(t-i)z_m^T(t-j)\} =$$

$$= \sum_{i=0}^{nc}\ \sum_{j=0}^{nc}\ c_i c_j R_z(j-i) = \sum_{\tau=-nc}^{nc} (\ \sum_{j=0}^{nc}\ c_j c_{j+\tau}) R_z(\tau) =$$

$$= \frac{1}{\lambda^2}\ \sum_{\tau=-nc}^{nc} r_v(\tau) R_z(\tau) \tag{37}$$

(In the following we will omit the facor $1/\lambda^2$ appearing in (37) since the IV estimates in (6) are invariant to scaling of the weighting matrix Q). Hence we can consistently estimate the optimal weighting matrix by $\hat{S}_m^{-1}$ where

$$\hat{S}_m = \sum_{\tau=-nc}^{nc} \hat{r}_v(\tau)\hat{R}_z(\tau) \tag{38}$$

where $\hat{r}_v(\tau)$ and $\hat{R}_z(\tau)$ are the sample covariances

$$\hat{r}_v(\tau) = \frac{1}{N} \sum_{t=\tau}^{N}\ v(t)v(t-\tau) = \hat{r}_v(-\tau) \tag{39}$$

$$\hat{R}_z(\tau) = \frac{1}{N} \sum_{t=\tau}^{N}\ z_m(t)\ z_m^T(t-\tau) = \hat{R}_z^T(-\tau) \tag{40}$$

Note that (38) provides a method for estimating S without explicit estimation of the $\{c_i\}$parameters. To estimate $\hat{r}_v(\tau)$ via (39) we need to compute (an estimate of)

$$v(t) = A(q^{-1})\ y(t) \tag{41}$$

An alternative way of computing $\hat{r}_v(\tau)$ follows from (41):

$$\hat{r}_v(\tau) = \sum_{i=0}^{na} \sum_{j=0}^{na} \hat{a}_i\ \hat{a}_j\ \hat{r}_y(\tau+i-j)\ . \tag{42}$$

Note that both (39)(41) and (42) require knowledge of the $\{a_i\}$ parameters. Since $A(q^{-1})$ is not known a-priori, we must use a multistep procedure. We first estimate $\{a_i\}$ using equation (6) with Q=I (and $G(q^{-1}) = 1$). The will be problem dependent, but generally m will be considerably larger than na; see [1][7]. This gives a consistent, although not efficient, estimate of the AR parameters. These estimates can now be used to compute $\hat{r}_v(\tau)$ via (39) and (41), or (42). Next we compute $\hat{S}_m$ (29) and use it in the OIV-1 procedure to get the final (asymptotically efficient) estimate of the AR parameters.

We can now summarize the proposed implementation of the OIV-1 estimator:

(i) Estimate $\{a_i\}$ by equation (6) with $Q = I$, $G(q^{-1}) = 1$ and $m > na$.

(ii) Compute $\hat{R}_z(\tau)$ (31) and $\hat{r}_v(\tau)$ by (39) and (41), or (42), using the $\{a_i\}$ estimates from step (i), and then compute $\hat{S}_m$ (38).

(iii) Compute the square-root $\hat{S}_m^{-1/2}$ of $\hat{S}_m^{-1}$; then solve equation (6) with $Q^{1/2} = \hat{S}_m^{-1/2}$ to obtain the final $\{a_i\}$ estimates.

Note that the computation of $\hat{S}_m$ via equation (40) does not guarantee that $\hat{S}_m$ will be a positive definite matrix. It may happen, therefore, that $\hat{S}_m^{-1/2}$ does not exist. This is unlikely to occur for large N, but is quite likely for small sample sizes (especially if C(z) has roots close to the unit circle).

The following is a procedure for handling the case where $\hat{S}_m$ is not positive definite. Let $\{\lambda_i\}_{i=1}^m$ be the ordered eigenvalues of $\hat{S}_m$, $\lambda_1 > \lambda_2 > \dots > \lambda_m$, and let $\{v_i\}_{i=1}^m$ be the corresponding

eigenvectors. Let

$$\lambda_k > \varepsilon \qquad k = 1,\ldots,n$$
$$\lambda_k < \varepsilon \qquad k = n \times 1, \ldots, m \tag{43}$$

with ε being a (small) positive number. Further, let $\mathscr{C}$ be the class of positive definite matrices with eigenvalues larger than or equal to ε. Then, according to Lemma A1 in appendix A, the Euclidean distance between $\hat{S}_m$ and the elements of $\mathscr{C}$ is minimal for the matrix $\tilde{S}_m$ given by

$$\tilde{S}_m = V\cdot[\mathrm{diag}(\lambda_1,\ldots,\lambda_n, \varepsilon,\ldots,\varepsilon)]\cdot V^T \tag{44}$$

where $V = [v_1,\ldots,v_m]$. We will use

$$\tilde{S}_m^{-1/2}=\mathrm{diag}(\frac{1}{\sqrt{\lambda_1}},\ldots,\frac{1}{\sqrt{\lambda_n}}, \frac{1}{\sqrt{\varepsilon}}, \ldots, \frac{1}{\sqrt{\varepsilon}})V^T \tag{45}$$

in (6) instead of $\hat{S}_m^{-1/2}$, which may not exist. Since $\tilde{S}_m$ must be a consistent estimate of $\lambda^2 S_m$, ε must go to zero as N tends to infinity. To guarantee consistency we may set $\varepsilon = 1/N^\beta$, $\beta > 0$. As $N \to \infty$ we will have $\tilde{S}_m \to \hat{S}_m$, where $\hat{S}_m$ is a consistent estimate of $\lambda^2 S_m$. Concerning the choice of β we note that the smaller ε , the smaller is the distance between $\tilde{S}_m$ and $\hat{S}_m$ cf. Lemma 1.1. However, too small an ε may lead to ill-conditioning problems. Thus ε should be chosen as a compromise between accuracy of the solution and numerical stability. Finally, note that if the estimated covariance matrix $\hat{S}_m$ happens to have negative eigenvalues then we may suspect that the $\{\hat{a}_i\}$ estimates obtained in Step (i) were poor. We may then wish to repeat Steps (ii)-(iii) using in Step (ii) the improved estimates of Step (iii).

4.2 THE OIV-2 ALGORITHM

The computation of the OIV-2 estimates requires the estimation of the $\{c_i\}$ parameters. There are, of course, many different ways in which this could be done. We consider here one such method based on factorization of the

MA spectrum [8]-[10].

Let $S_v(z)$ denote the spectral density fucntion of v(t), (2), (41). We have

$$S_v(z) \triangleq \sum_{k=-nc}^{nc} r_v(k)z^{-k} = \lambda^2 C(z)C(z^{-1}) \tag{46}$$

where $r_v(k)$ denotes the covariance of v(t) at lag k (35). In other words, the C(z) polynomial is the spectral factor of the spectrum of v(t). This suggests the following procedure for estimating the $\{c_i\}$ parameters:

(a) Estimate the $\{a_i\}$ parameters using (6) with Q=I, $G(q^{-1}) = 1$, $m > na$.

(b) Compute the sample covariances $\hat{r}_v(k)$, k=0,...,nc, using (39) and (41) or (42).

(c) Perform spectral factorization of $\hat{S}_v(z) = \sum_{k=-nc}^{nc} \hat{r}_v(k)z^{-k}$ to obtain $\{\hat{c}_i\}$.

Note that the sample covariance sequence $\{\hat{r}_v(0),\ldots,\hat{r}_v(nc),0,0,\ldots\}$ is not guaranteed to be positive definite. Thus, $\hat{S}_v(z)$ may not be factorizable. This may happen in the small sample case, especially when C(z) has roots close to the unit circle. However, note that OIV-2 requires an estimate of $\hat{C}^2(q^{-1})$ rather than of $\hat{C}^2(q^{-1})$ We can always obtain a consistent estimate of $\hat{C}^2(q^{-1})$ by factoring $\hat{S}_v^2(z)$, since

$$\hat{S}_v^2(e^{j\omega})=\hat{C}^2(e^{j\omega})\hat{C}^2(e^{-j\omega}) \geq 0, \text{ for all } \omega \tag{47}$$

even though $\hat{S}_v(e^{j\omega})$ may be negative for some values of ω.

We can now summarize the proposed implementation of OIV-2:

(i) Estimate $\hat{C}(q^{-1})$ using the spectral factorization method described above. Let $\hat{G}(q^{-1}) = 1/\hat{C}^2(q^{-1})$.

(ii) Estimate the AR parameters using equation (6) with $Q = I$, $m = na$ and $\hat{G}(q^{-1})$ from step (i).

Note that it is possible to iterate this procedure by using the AR parameters obtained in step (ii) to improve the estimate of $C(q^{-1})$, by repeating step (i) (the factorization method) with the new $\{\hat{a}_i\}$ parameters.

4.3 THE OIV-3 ALGORITHM

The computation of the OIV-3 estimates is very similar to that of OIV-2. The only difference is that $\hat{G}(q^{-1}) = \hat{A}(q^{-1})/\hat{C}^2(q^{-1})$ where $\hat{A}(q^{-1})$ is the current estimate of $A(q^{-1})$ (obtained from step (a) in the first iteration of the algorithm, or from the previous step (ii) in the case of re-iteration).

4.4 COMPUTATIONAL REQUIREMENTS

The following is a brief summary of the number of arithmetic operations (i.e., multiplies and adds) required by each of the algorithms described above.

OIV-1:

Step (i): requires approximately $\sim(m+na)N$ operations to compute the sample covariances and $\sim m^3$ operations to solve for the initial estimate (solutions requiring only $\sim m^2$ operations are also possible if the Toeplitz structure of the Yule-Walker equation is used). Step (ii): requires $\sim(na)^2 \cdot nc$ operations to evaluate $\hat{r}_v(\tau)$ using (42), or $\sim(na+nc)N$ operations using (39), (41). The computation of $\hat{S}_m$ requires $\sim nc \cdot m^2$ operations. Step (iii): requires $\sim 3m^3$ operations. A recursive QR algorithm which appears to be useful for solving (6) is presented in Appendix B.

OIV-2:

Step (a): requires $\sim(m+na)N + m^3$ operations, as in the case of OIV-1. Steps (b): involves the computation of $\hat{r}_v(\tau)$ which requires either $\sim(na)^2 nc$ or $\sim(na+nc)N$ operations. Step (c): computational requirements will depend on

the particular factorization technique. Step (ii): requires $\sim 2(nc+na)N + (na)^3$ operations.

OIV-3:

Steps (a)-(c) -- same as OIV-2. Step (ii) -- same as step (iii) of OIV-1, with the addition of (2nc+na)N operations to perform the pre-filteing.

In summary:

OIV-1: $(m+na)N+4m^3+nc\cdot m^2$ (or $(m+2na+nc)N+4m^3+ncm^2$)

OIV-2: $(m+2na+2nc)N+m^3+na^2(nc+na)$ (or $(m+4na+4nc)N+m^3+na^3$)

OIV-3: $(m+2na+2nc)N+na^2nc$ (or $(m+3na+3nc)N+4m^3$)

Note also that re-iteration of OIV-1 does not require much computation since the sample covariances need to be computed only once. Iteration of OIV-2 and OIV-3 is more costly since the data need to be refiltered and some sample covariances recomputed at each iteration.

5. NUMERICAL EXAMPLES

In this section we present some selected results of computer simulations which illustrate the behavior of the OIV algorithms discussed earlier. Tables 1-10 summarize results based on 100 independent Monte-Carlo runs performed for each of the test cases described below. Each of the tables contains the means and standard deviations (as well as mean-squared-errors) of the AR parameter estimates obtained by applying the MYW, OIV-1, OIV-2 and OIV-3 algorithms to simulated data. The OIV algorithms were used with different values of m and iterated three times.

Note that OIV-2 was run for values of m different from m=na. The asymptotic theory shows that OIV-2 is optimal only for m=na, not for m > na. However, in the finite data case we found that increasing m tended to make the algorithm more robust by reducing the probability of singularity of the matrix

which needs to be inverted. In the first two cases the data were the sum of a second order autoregressive process and white noise:

$$y(t) = x(t) + n(t) , \tag{48}$$

where

$$x(t) = -a_1 x(t-1) - a_2 x(t-2) + w(t) \tag{49}$$

and where $w(t)$ and $n(t)$ are mutually uncorrelated white noise processes whose variances were chosen to give the desired signal-to-noise ratio ($SNR = Var\{x(t)\}/Var\{n(t)\}$) . As is well-known, $y(t)$ has an equivalent ARMA (2,2) representation. The zeroes of the MA part can be shown to be farther away from the unit circle than the zeroes of the AR part. As the SNR decreases, the MA zeroes approach the AR zeroes.

<u>Case 1:</u> Narrowband, high SNR

$$A(z) = 1 - 1.4z^{-1} + 0.95z^{-2} , \text{ (zeroes at } .975 \cdot e^{\pm j44.1^0}),$$

$$SNR = 20 \text{ dB}, N = 4096$$

The MA polynomial of the equivalent ARMA representation of this process is

$$C(z) = 1 - 0.3155z^{-1} + 0.1233z^{-2} . , \text{ (zeroes at } .351 \cdot e^{\pm j63.3^0})$$

The results are summarized in tables 1 and 2. In this high SNR case the experimental results are very close to the asymptotic bounds.

<u>Case 2:</u> Narrowband, low SNR
As in case 1, with SNR = 0 dB

$$C(z) = 1 - 1.20955z^{-1} + 0.726837z^{-2}, \text{ (zeroes at } .853 \cdot e^{\pm j44.8^0})$$

The results are summarized in tables 3 and 4.

TABLE 1: Experimental and Theoretical Estimation Accuracy for Case 1, Parameter a_1

m			2	4	10	40
MYW	Mean ± std.dev.		-1.40±.0.0531	-1.40±.00530	-1.40±.00530	-1.40±.00611
	mse		0.00542	.00541	.00537	.00619
	Theoretical std.dev.			0.00533		
OIV-1	Iteration 1	Mean ± std.dev.	--	-1.40±.00538	-1.40±.00528	-1.40±.00537
		mse	--	.00549	.00539	.00548
	Iteration 2	Mean ± std.dev.	--	-1.40±.00545	-1.40±.00535	-1.40±.00534
		mse	--	.00556	.00546	.00545
	Iteration 3	Mean ± std.dev.	--	-1.40±.00544	-1.40±.00541	-1.40±.00531
		mse	--	.00555	.00552	.00542
	Theoretical std.dev.			0.00532		
OIV-2	Iteration 1	Mean ± std.dev.	-1.40±0.00540	-1.40±.00547	-1.40±.00552	-1.40±.00631
		mse	0.00550	.00558	.00559	.00638
	Iteration 2	Mean ± std.dev.	-1.40±0.00547	-1.40±.00538	-1.40±.00544	-1.40±.00632
		mse	0.00557	.00550	.00551	.00639
	Iteration 3	Mean ± std.dev.	-1.40±0.00530	-1.40±.00540	-1.40±.00548	-1.40±.00632
		mse	0.00541	.00551	.00555	.00639
	Theoretical std.dev.		0.00532		--	----
OIV-3	Iteration 1	Mean ± std.dev.	--	-1.40±.0135	-1.40±.0104	-1.40±.00607
		mse	--	.0139	.0106	.00630
	Iteration 2	Mean ± std.dev.	--	-1.40±.0132	-1.40±.0103	-1.40±.00623
		mse	--	.0133	.0103	.00633
	Iteration 3	Mean ± std.dev.	--	-1.40±0140	-1.40±.0106	-1.40±.00618
		mse	--	.0141	.0106	.00629
	Theoretical std.dev.			0.0122		

TABLE 2: Experimental and Theoretical Estimation Accuracy for Case 1, Parameter a_2

m			2	4	10	40
MYW	Mean ± std.dev.		.949±.00476	.9491±.004792	.9492±.004974	.9492±.005823
	mse		0.004863	.004882	.005031	.005878
	Theoretical std.dev.			0.00508		
OIV-1	Iteration 1	Mean ± std.dev.	--	.9490±.004816	.9490±.004784	.9490±.004792
		mse	--	.004916	.004883	.004894
	Iteration 2	Mean ± std.dev.	--	.9490±.004856	.9490±.004800	.9490±.004800
		mse	--	.004955	.004800	.004902
	Iteration 3	Mean ± std.dev.	--	.9490±.004840	.9490±.004800	.9490±.004816
		mse	--	.004939	.004899	.004918
	Theoretical std.dev.			0.00506		
OIV-2	Iteration 1	Mean ± std.dev.	.949±.004981	.9490±.005058	.9493±.005164	.9493±.006057
		mse	0.005096	.005154	.005217	.006101
	Iteration 2	Mean ± std.dev.	.949±0.004966	.9490±.004989	.9493±.005134	.9493±.006082
		mse	0.005080	.005086	.005188	.006126
	Iteration 3	Mean ± std.dev.	.949±0.004958	.9490±.005012	.9493±.005134	.9493±.006044
		mse	0.005073	.005109	.005188	.006089
	Theoretical std.dev.		0.00506	--	--	----
OIV-3	Iteration 1	Mean ± std.dev.	--	.9467±.01227	.9480±.007988	.9480±.005559
		mse	--	.01270	.008243	.005912
	Iteration 2	Mean ± std.dev.	--	.9493±.01229	.9494±.008136	.9487±.005587
		mse	--	.01231	.008155	.005739
	Iteration 3	Mean ± std.dev.	--	.9493±.01285	.9495±.008342	.948±.005614
		mse	--	.01286	.008360	.005771
	Theoretical std.dev.			0.0117		

TABLE 3: Experimental and Theoretical Estimation Accuracy for Case 2, Parameter a_1

m			2	4	10	40
MYW	Mean ± std.dev.		1.729±2.597	-.926±.390	-.811±.275	-.5521±.1883
	mse		2.618	.614	.650	.8686
	Theoretical std.dev.			0.0130		
OIV-1	Iteration 1	Mean ± std.dev.	--	-1.16±.399	-1.16±.307	-.9154±.2834
		mse	--	.468	.388	.5614
	Iteration 2	Mean ± std.dev.	--	-1.21±.397	-1.26±.301	-1.152±.2800
		mse	--	.442	.333	.3740
	Iteration 3	Mean ± std.dev.	--	-1.22±.391	-1.28±.288	-1.266±.2372
		mse	--	.429	.311	.2722
	Theoretical std.dev.			0.0127		
OIV-2	Iteration 1	Mean ± std.dev.	-0.8098±2.269	-1.27±.335	-1.29±.270	-1.192±.2370
		mse	2.344	.356	.292	.3154
	Iteration 2	Mean ± std.dev.	-1.096±.9459	-1.30±.317	-1.34±.275	-1.375±.1276
		mse	.9935	.334	.281	.1301
	Iteration 3	Mean ± std.dev.	-0.9804±1.240	-1.32±.315	-1.34±.346	-1.405±.1195
		mse	1.309	.327	.351	.1197
	Theoretical std.dev.		0.00674	--	--	--
OIV-3	Iteration 1	Mean ± std.dev.	--	-1.15±.423	-1.19±.296	-.9724±.2806
		mse	--	.493	.362	.5115
	Iteration 2	Mean ± std.dev.	--	-1.21±.371	-1.291±.2847	-1.198±.2507
		mse	--	.417	.3047	.3222
	Iteration 3	Mean ± std.dev.	--	-1.27±.328	-1.313±.2953	-1.281±.2029
		mse	--	.354	.3079	.2351
	Theoretical std.dev.			.0156		

TABLE 4: Experimental and Theoretical Estimation Accuracy for Case 2, Parameter a_2

m			2	4	10	40
MYW	Mean ± std.dev.		.9279±2.672	.5293±.3846	.4763±.2238	.2484±.1413
	mse		2.672	.5700	.5239	.7157
	Theoretical std.dev.		--	0.0211		
OIV-1	Iteration 1	Mean ± std.dev.	--	.6980±.4347	.7543±.2424	.5358±.2247
		mse	--	.5025	.3115	.4712
	Iteration 2	Mean ± std.dev.	--	.7405±.4275	.8369±.2354	.7375±.2229
		mse	--	.4761	.2611	.3079
	Iteration 3	Mean ± std.dev.	--	.7541±.4232	.8563±.2209	.8342±.1828
		mse	--	.4664	.2399	.2164
	Theoretical std.dev.			0.0194		
OIV-2	Iteration 1	Mean ± std.dev.	.7778±.6499	.8279±.3295	.8676±.2074	.7636±.1945
		mse	.6723	.3515	.2231	.2694
	Iteration 2	Mean ± std.dev.	.8574±.7826	.8695±.2966	.9271±.1424	.9283±.08034
		mse	.7880	.3074	.1442	.08321
	Iteration 3	Mean ± std.dev.	.8611±.4633	.8934±.2389	.9440±.07594	.9504±.03850
		mse	.4717	.2455	.07618	.03850
	Theoretical std.dev.		0.0067	--	--	--
OIV-3	Iteration 1	Mean ± std.dev.	--	.7661±.2853	.7815±.2326	.5844±.2260
		mse	--	.3395	.2872	.4299
	Iteration 2	Mean ± std.dev.	--	.8008±.3085	.8603±.2122	.7783±.2025
		mse	--	.3426	.2304	.2655
	Iteration 3	Mean ± std.dev.	--	.8337±.2615	.8904±.2102	.8443±.1508
		mse	--	.2862	.2185	.1841
	Theoretical std.dev.			0.0157		

TABLE 5: Experimental and Theoretical Estimation Accuracy for Case 3, Parameter a_1

m			2	4	10	40
MYW	Mean ± std.dev.		.3401±11.98	-.9641±.4967	-.8148±.2602	-.3607±.1214
	mse		12.12	.7307	.7329	1.146
	Theoretical std.dev.			0.547		
OIV-1	Iteration 1	Mean ± std.dev.		-1.285±.6152	-1.452±.06578	-1.302±.09564
		mse		.6517	.08153	.2202
	Iteration 2	Mean ± std.dev.		-1.320±.6188	-1.491±.04059	-1.478±.03514
		mse		.6443	.04164	.04164
	Iteration 3	Mean ± std.dev.		-1.333±.6047	-1.491±.04180	-1.483±.03768
		mse		.6272	.04276	.04137
	Theoretical std.dev.			0.491		
OIV-2	Iteration 1	Mean ± std.dev.		-1.454±.2030	-1.484±.04267	-1.356±.1225
		mse		.2082	.04562	.1891
	Iteration 2	Mean ± std.dev.		-1.475±.2055	-1.496±.05307	-1.505±.05697
		mse		.2070	.05325	.05724
	Iteration 3	Mean ± std.dev.		-1.474±.2226	-1.506±.05888	-1.514±.06048
		mse		.2243	.05916	.06199
	Theoretical std.dev.		0.0139	--	--	--
OIV-3	Iteration 1	Mean ± std.dev.	-.9022±7.139	-1.429±.2372	-1.445±./07617	-1.152±.1534
		mse	7.164	.2476	.09402	.3800
	Iteration 2	Mean ± std.dev.	-1.569±1.134	-1.459±.2476	-1.466±.06375	-1.430±.1044
		mse	1.136	.2510	.07213	.1258
	Iteration 3	Mean ± std.dev.	-1.492±0.1007	-1.467±.2483	-1.466±.09093	-1.423±.089716
		mse	0.1010	.2505	.09710	.1166
	Theoretical std.dev.			0.0160		

TABLE 6: Experimental and Theoretical Estimation Accuracy for Case 3, Parameter a_2

m			2	4	10	40
MYW	Mean ± std.dev.		-.5767±9.21	.3323±.3235	.2114±.1877	-.07178±.1006
	mse		9.299	.4897	.5234	.7783
	Theoretical std.dev.			0.335		
OIV-1	Iteration 1	Mean ± std.dev.		.5733±.3739	.6711±.04926	.5656±.07516
		mse		.3948	.05712	.1540
	Iteration 2	Mean ± std.dev.		.5961±.3756	.6953±.03093	.6889±.02725
		mse		.3897	.03129	.02942
	Iteration 3	Mean ± std.dev.		.6051±.3654	.6956±.03173	.6956±.02699
		mse		.3775	.03203	.02734
	Theoretical std.dev.			0.301		
OIV-2	Iteration 1	Mean ± std.dev.	-0.7038±13.22	.6674±.1260	.6835±.03746	.5894±.09895
		mse	13.29	.1302	.04093	.1484
	Iteration 2	Mean ± std.dev.	0.7563±.6148	.6845±.1222	.7003±.04660	.7137±.04116
		mse	.6173	.1232	.04660	.04337
	Iteration 3	Mean ± std.dev.	0.6952±.08419	.6912±.1350	.7095±.05817	.7373±.06200
		mse	0.08433	.1352	.05893	.07234
	Theoretical std.dev.		0.0138			
OIV-3	Iteration 1	Mean ± std.dev.		.6626±.1356	.6610±.05672	.4685±.1128
		mse		.1407	.06885	.2575
	Iteration 2	Mean ± std.dev.		.6771±.1479	.6815±.04338	.6543±.06464
		mse		.1497	.04718	.07915
	Iteration 3	Mean ± std.dev.		.6914±.1523	.6819±.06095	.6598±.05870
		mse		.1526	.06359	.07113
	Theoretical std.dev.			0.0185		

TABLE 7: Experimental and Theoretical Estimation Accuracy for Case 4, Parameter a_1

		m	2	4	10	40
MYW		Mean ± std.dev.	-1.687±.007969	-1.687±.005607	-1.687±.005902	-1.687±.006709
		mse	.008070	.005721	.005983	.006749
		Theoretical std.dev.				
OIV-1	Iteration 1	Mean ± std.dev.		-1.687±.005716	-1.687±.005296	-1.686±.009568
		mse		.00582	.005359	.009733
	Iteration 2	Mean ± std.dev.		-1.687±.005716	-1.687±.005325	-1.688±.01345
		mse		.005822	.005387	.01345
	Iteration 3	Mean ± std.dev.		-1.687±.005716	-1.687±.005325	-1.689±.01890
		mse		.005822	.005387	.01890
	Theoretical std.dev.					
OIV-2	Iteration 1	Mean ± std.dev.	-1.688±.005354	-1.688±.005325	-1.688±.005607	-1.688±.006522
		mse	.005397	.005371	.005651	.006550
	Iteration 2	Mean ± std.dev.	-1.687±.005552	-1.688±.005468	-1.688±.00524	-1.688±.006709
		mse	.005601	.005512	.005566	.005736
	Iteration 3	Mean ± std.dev.	-1.688±.005383	-1.688±.005635	-1.688±.005552	-1.688±.0066663
		mse	0.005423	.005677	.005594	.006690
	Theoretical std.dev.			--	--	--
OIV-3	Iteration 1	Mean ± std.dev.	--	-1.687±.01697	-1.687±.009324	-1.686±.006355
		mse		.01701	.009392	.006620
	Iteration 2	Mean ± std.dev.	--	-1.692±.01869	-1.689±.009072	-1.687±.005980
		mse		.01901	.009116	.006050
	Iteration 3	Mean ± std.dev.		-1.692±.02019	-1.689±.009854	-1.687±.006132
		mse		.02046	.009880	.006221
	Theoretical std.dev.					

TABLE 8: Experimental and Theoretical Estimation Accuracy for Case 4, Parameter a_2

m			2	4	10	40
MYW	Mean ± std.dev.		.9488±.007186	.9487±.006604	.9488±.006789	.9491±.007339
	mse		.007325	.006738	.006885	.007399
	Theoretical std.dev.					
OIV-1	Iteration 1	Mean ± std.dev.		.9487±.006703	.9488±.006391	.9467±.01466
		mse		.006836	.006499	.01503
	Iteration 2	Mean ± std.dev.		.9487±.006691	.9488±.006373	.9497±.01053
		mse		.006825	.006480	.01053
	Iteration 3	Mean ± std.dev.		.9487±.006674	.9488±.006336	.9481±.01965
		mse		.006808	.006445	.01974
	Theoretical std.dev.					
OIV-2	Iteration 1	Mean ± std.dev.	.9489±.006385	.9489±.006409	.9491±.006616	.9492±.007355
		mse	.006478	.006498	.006684	.007395
	Iteration 2	Mean ± std.dev.	.9489±.006445	.9489±.006415	.9491±.006593	.9492±.007387
		mse	.006544	.00650	.006658	.007426
	Iteration 3	Mean ± std.dev.	.9490±.006415	.9489±.006397	.9491±.006622	.9492±.007387
		mse	.006500	.006483	.006687	.007426
	Theoretical std.dev.			--	--	--
OIV-3	Iteration 1	Mean ± std.dev.		.9480±.02086	.9483±.01044	.9479±.006935
		mse		.02095	.01057	.007243
	Iteration 2	Mean ± std.dev.		.9531±.02286	.9506±01010	.9489±.006760
		mse		.02307	.01011	.006855
	Iteration 3	Mean ± std.dev.		.9536±.02681	.9506±.01087	.9488±.006845
		mse		.02705	.01089	.006950
	Theoretical std.dev.					

bi-diagonalization and a QR algorithm. Solutions corresponding to singular values less than 10^{-8} times the largest singualr value were set to zero (in effect decreasing the assumed rank of the MYW equations and producing the minimum-norm solution of the underdetermined set of equations).

OIV-1: The algorithm was implemented as described in section 4 and appears to be quite robust. Since our simulations involved relatively long data records we did not encounter problems with $\hat{S}_m$ being non-positive definite. Thus, we did not have to use the procedure described in (43)-(45). In fact we computed $\hat{S}_m^{-1/2}$ by the Levinson-Durbin algorithm, applied to the first column of $\hat{S}_m$. We used equations (39)(41) to estimate $\hat{r}_v(\tau)$.

OIV-2: The factorization of $\hat{S}_v(z)$ was performed by computing the roots of $z^{nc}\hat{S}_v(z)$. All the roots outside the unit circle were reflected inside the unit circle, and the complete set of roots was then used to compute $\hat{C}^2(z)$. In this case we noticed that the filtering operation (by $G(q^{-1}) = 1/\hat{C}^2(q^{-1})$) introduced a transient which needed to be eliminated. To limit the duration of the transient we "contracted" the roots of the polynomial $\hat{C}^2(z)$ by replacing $\hat{C}^2(z)=1+\hat{c}_1'z^{-1}+\ldots+\hat{c}_{nc}'z^{-2nc}$ by $\hat{C}^2(z/\eta)=1+\hat{c}_1'\eta z^{-1}+\ldots+\hat{c}_{2nc}'\eta^{2nc}z^{-2nc}$, where $\eta = 0.99$. By construction, the roots of $\hat{C}^2(z)$ have maximum modulus of 1. To eliminate the effects of transients in $(\hat{G}(q^{-1})y(t))$, the first 200 samples of the filtered data were discarded.

OIV-3: Implementation was very similar to OIV-2.

6. CONCLUSIONS

We presented several multistep implementations of optimal instrumental variable algorthms for estimating the AR parameters of an ARMA process. These algorithms were shown to provide asymptotically efficient estimates of the AR parameters at a modest computational cost, compared to methods such as the Maximum Likelihood Estimator. The OIV algorithms are useful in situations where the MYW method does not provide accurate estimates (e.g., for ARMA processes with zeroes near the unit circle). The performance of the proposed algorithms was illustrated by selected numerical examples.

ACKNOWLEDGMENTS

The authors gratefully acknowledge the contribution of Dr. J. O. Smith who provided the simulation results for this work.

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APPENDIX A. THE BEST POSITIVE-DEFINITE APPROXIMATION OF A SYMMETRIC MATRIX

Let A be a mxm symmetric matrix. Let $\lambda_1 > \lambda_2 > \dots \lambda_m$ be its eigenvalues and $v_1,\dots,v_m$ be the corresponding eigenvectors. We have the following result, which is a slight modification of a similar result given in [ ].

Lemma A1. Let be the class of positive definite mxm matrices with eigenvalues larger than or equal to a given (small) positive number ε. Then

$$\inf_{B\varepsilon} \|A-B\| = [(\lambda_{n+1}-\varepsilon)^2 + \dots + (\lambda_m-\varepsilon)^2]^{1/2} \tag{A.1}$$

where $\|A\| = [\mathrm{tr}AA^T]^{1/2} = [\sum_i \sum_j a_{ij}^2]^{1/2}$ denotes the Euclidean norm, and $\lambda_{n+1},\dots,\lambda_m$ are the eigenvalues of A that are smaller than ε, that is

$$\begin{aligned} \lambda_k &> \varepsilon \qquad k=1,\dots,n \\ \lambda_k &< \varepsilon \qquad k=n+1,\dots,m \end{aligned} \tag{A.2}$$

Furthermore, the infimum is attained for

$$B = V \cdot \begin{bmatrix} \lambda_1 & & & & & 0 \\ & \ddots & & & & \\ & & \lambda_n & & & \\ & & & \varepsilon & & \\ & & & & \ddots & \\ 0 & & & & & \varepsilon \end{bmatrix} \cdot V^T \qquad \text{(A.3)}$$

with $V = [v_1, \ldots, v_m]$.

<u>Proof</u>: We have

$$\|A-B\|^2 = \|V^T(A-B)V\|^2 = \sum_{i=1}^{m} (\lambda_i - c_{ii})^2 + \sum_{\substack{i,j=1 \\ i \neq j}}^{m} c_{ij}^2$$

where c_{ij} is the i,j-element of

$$C = V^T B\ V \qquad \text{(A.4)}$$

Clearly C has the same eigenvalues as B. Thus we can write

$$\|A-B\|^2 \geq \sum_{i=1}^{m} (\lambda_i - c_{ii})^2 \geq \sum_{i=n+1}^{m} (\lambda_i - c_{ii})^2 \geq \sum_{i=n+1}^{m} (\lambda_i - \varepsilon)^2$$

where the equalities hold if

$$c_{ij}=0 \quad i \neq j; \; c_{ii}=\lambda_i \quad i=1,\ldots,n \quad c_{ii}=\varepsilon \quad i=n+1,\ldots,m \qquad \text{(A.5)}$$

By inserting (A.5) in (A.4) we readily obtain (A.3).

APPENDIX B: A RECURSIVE QR ALGORITHM FOR SOLVING (6)

Let us rewrite equation (6) as

$$L_m \hat{\theta}_m = \ell_m \qquad \text{(B.1)}$$

where

$$L_m = \hat{S}_m^{-1/2} \sum_{t=1}^{N} z_m(t)\phi(t)^T \tag{B.2}$$

$$\ell_m = \hat{S}_m^{-1/2} \sum_{t=1}^{N} z_m(t)y(t) \tag{B.3}$$

Let L_m be factored as

$$L_m = O_m T_m \tag{B.4}$$

where

$$O_m = \text{An orthogonal matrix} \tag{B.5}$$

$$T_m = \text{An upper triangular matrix} \tag{B.6}$$

Then $\hat{\theta}_m$ can be computed by back-substitution from

$$T_m \hat{\theta}_m = O_m^T \ell_m \tag{B.7}$$

Consider now the situation for (m+1). Determine first α and β in

$$\hat{S}_{m+1}^{-1/2} = \begin{bmatrix} \hat{S}^{-1/2} & 0 \\ \beta^T & \alpha \end{bmatrix} \tag{B.8}$$

and then

$$L_{m+1} = \hat{S}_{m+1}^{-1/2} R_{m+1} = \begin{bmatrix} L_m \\ \gamma^T \end{bmatrix} \tag{B.9}$$

We have

$$L_{m+1} = \begin{bmatrix} O_m & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} T_m \\ \gamma \end{bmatrix} \tag{B.10}$$

So the problem of factorizing L_{m+1} reduces to the factorization of $\begin{bmatrix} T_m \\ \gamma^T \end{bmatrix}$.

In this last matrix only the last row γ^T needed to be made zero. The computations needed are clearly simpler than if the matrix would have been full. Let $\bar{O}_{m+1}$ be an orthogonal matrix such that

$$\bar{O}_{m+1} \begin{bmatrix} T_m \\ \gamma^T \end{bmatrix} = \text{triangular} \triangleq T_{m+1} , \tag{B.11}$$

or,

$$L_{m+1} = \underbrace{\begin{bmatrix} O_m & 0 \\ 0 & 1 \end{bmatrix} \bar{O}^T_{m+1}}_{\bar{O}_{m+1}} T_{m+1} = O_{m+1} T_{m+1} . \tag{B.12}$$

Finally, we have

$$\ell_{m+1} = \hat{S}^{-1/2}_{m+1} \sum_{t=1}^{N} z_m(t)\phi(t)^T = \begin{bmatrix} \ell_m \\ \rho \end{bmatrix} \tag{B.13}$$

The estimate $\hat{\theta}_{m+1}$ is computed from

$$T_{m+1}\hat{\theta}_{m+1} = O^T_{m+1}\ell_{m+1} = \bar{O}_{m+1} \begin{bmatrix} O^T_m & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \ell_m \\ \rho \end{bmatrix} \tag{B.14}$$

or,

$$T_{m+1}\hat{\theta}_{m+1} = \bar{O}_{m+1} \begin{bmatrix} O^T_m \ell_m \\ \rho \end{bmatrix} \tag{B.15}$$

APPENDIX G

AN APPROXIMATE MAXIMUM LIKELIHOOD APPROACH TO ARMA SPECTRAL ESTIMATION

AN APPROXIMATE MAXIMUM LIKELIHOOD APPROACH TO ARMA SPECTRAL ESTIMATION

Petre Stoica, Benjamin Friedlander and Torsten Söderström

ABSTRACT

A three-step approximate maximum likelihood method for ARMA spectral estimation is derived, based on an idea due to Walker. The asymptotic properties of the proposed estimator are investigated and an explicit expression for its asymptotic covariance matrix is presented. The estimator provides the asymptotic accuracy of a maximum likelihood technique, at a modest computational cost.

The work of B. Friedlander was supported by the Army Research Office under contract No. DAAG29-83-C-0027. P. Stoica is with Facultatea de Automatica, Institutul Politehnic Bucuresti, Splaiul Independentei 313, Sector 6, R-77 206 Bucharest, Romania. B. Friedlander is with Systems Control Technology, Inc., 1801 Page Mill Rd., Palo Alto, CA 94304, USA. T. Söderström is with Department of Automatic Control and Systems Analysis, Institute of Technology, Uppsala University, P.O. Box 534, S-751 21 Uppsala, Sweden.

1. INTRODUCTION

Autoregressive moving-average (ARMA) spectral estimation is a topic of considerable interest in engineering, econometrics, biometrics, statistics and other areas [1]-[8], [16]-[22]. Many different methods have been proposed for estimating the ARMA spectrum, including: (i) Optimization-based procedures such as the maximum likelihood (ML) method and various nonlinear least-squares techniques [1],[3],[4],[8],[21],[22]. These methods tend to be computationally intensive and have inherent difficulties due to possible convergence to local minima. (ii) Techniques based on the Yule-Walker method and its many variations [1],[5]-[7],[12],[13], [23]-[25]. These methods involve the solution of a linear set of equations and do not suffer from convergence to false minima. However, the accuracy of the estimates may be poor unless special measures are taken, such as increasing the number of equations [7],[12], increasing the order of the model [5]-[7] or choosing an optimal weighting matrix [23]-[25].

In this paper we develop an estimation technique which combines the computational simplicity of the Yule-Walker based methods with the accuracy of ML techniques. The proposed estimator is based on an idea due to Walker [10], [11], involving large-sample approximate ML estimation of the covariances of the observed ARMA process. These covariance estimates are then used in a Yule-Walker based procedure to obtain approximate ML estimates of the ARMA spectral parameters.

The spectral estimation method proposed here is more general than the related method in [11] (See also [9], [26], [28]). Walker considered the estimation of correlation coefficients instead of covariances and his results can not be used in a straightforward manner for ARMA spectrum estimation. We introduce here an estimation technique based on a maximum likelihood approach similar to, but simpler than, the approach used in [11] (see also [33]). A large-sample ML method is introduced and its accuracy properties are established in a general setting. This general analysis is believed to be interesting in its own right, and could be used to obtain large-sample ML estimates for various estiation problem besides the one considered here (see [25]). The ARMA spectral estimator derived here is shown to be asymptotically

efficient. The proof of its efficiency is a key contribution of this paper.

The outline of the paper is as follows: In section 2 we present the spectral model considered in this paper and discuss some alternative parametrizations. A large sample approximate solution to a general maximum likelihood estimation problem is derived in section 3 and its accuracy properties are discussed. In section 4 we specialize this approximate ML approach to the ARMA spectral estimation problem. A specific estimation algorithm is proposed. The asymptotic accuracy properties of the proposed estimator are discussed in section 5, and its asymptotic error covariance is compared to the Cramér-Rao lower bound in section 6.

2. THE SPECTRAL MODEL

Consider the following ARMA process of order (na,nc)

$$A(q^{-1})\, y(t) = C(q^{-1})\, e(t) \,, \tag{2.1}$$

where

$e(t)$ = white noise process with zero mean and variance λ^2 ,

$A(q^{-1}) = 1 + a_1 q^{-1} + \ldots + a_{na} q^{-na}$,

$C(q^{-1}) = 1 + c_1 q^{-1} + \ldots + c_{nc} q^{-nc}$,

q^{-1} = unit delay operator $\quad (q^{-1}y(t)=y(t-1))$.

The following standard assumptions are made:

A1: $A(z) \cdot C(z) = 0 \Rightarrow |z| > 1$

In other words, the ARMA representation (2.1) is stable and invertible. This is not a restrictive assumption, cf. the spectral factorization theorem [29]. We note, however, that there are some cases of interest where A1 does not hold. For example, the sinusoids-in-noise process can be described by an ARMA model (2.1) with $A(z) = C(z)$ and $A(z) = 0 \Rightarrow |z| = 1$, [1]-[3]. As we

shall explain later, the method of this paper does not extend to such "degenerate" ARMA processes.

A2: $a_{na} \cdot c_{nc} \neq 0$ and $\{A(z), C(z)\}$ are coprime polynomials.

In other words, (na,nc) are the minimal orders of the ARMA model (2.1). In the following we assume for simplicity that (na,nc) are given.

Next we introduce the following notation:

$$r_k = E\{y(t)\ y(t-k)\} = \text{the covariance of } y(t) \text{ at lag } k, \tag{2.2a}$$

$$\phi(z) = \sum_{k=-\infty}^{\infty} r_k z^{-k} = \text{the spectral density of } y(t). \tag{2.2b}$$

In (2.2) $E\{\cdot\}$ denotes the expectation operator and z is a complex variable.

It is well known that

$$\phi(z) = \lambda^2 \frac{C(z)C(z^{-1})}{A(z)A(z^{-1})} . \tag{2.3}$$

Thus, $\phi(z)$ could be parametrized via $\{a_i\}$, $\{c_j\}$ and λ^2. The statistically efficient estimation of these parameters is not an easy task (even though asymptotically efficient estimates of $\{a_i\}$ can be obtained by using only linear operations [23]).

In this paper we parametrize $\phi(z)$ by the covariances $\{r_k\ ,\ k=0,\ldots,na+nc\}$. These covariances uniquely define $\phi(z)$. The sequence $\{r_k\}$ satisfies the well-known Yule-Walker equations:

$$r_k + a_1 r_{k-1} + \ldots + a_{na}\ r_{k-na} = 0\ , \qquad k > nc + 1. \tag{2.4}$$

Introduce the notation

$$b_k \triangleq E\{A(q^{-1})y(t)\ A(q^{-1})y(t-k)\} = \sum_{i=0}^{na} \sum_{j=0}^{na} a_i a_j\ r_{k+j-i}\ , \tag{2.5}$$

where $a_0 = 1$. It then readily follows from (2.1), (2.3) and (2.5) that

$$\phi(z) = \frac{\sum_{k=-nc}^{nc} b_k z^{-k}}{A(z)A(z^{-1})} \quad . \tag{2.6}$$

Note that the numerator of (2.6) is a function of $\{r_k, k=0,\ldots,na+nc\}$. Next note that the coefficients $\{a_i, i=1,\ldots,na\}$ can be uniquely determined from $\{r_k, k=0,\ldots,na+nc\}$ by using (2.4). This is possible since under the assumptions A1 and A2 the matrix

$$R \triangleq \begin{bmatrix} r_{nc} & \ldots & r_{nc+1-na} \\ r_{nc+1} & \ldots & r_{nc+2-na} \\ \vdots & & \vdots \\ r_{nc+na-1} & \ldots & r_{nc} \end{bmatrix} , \tag{2.7}$$

arising from the system of equations (2.4), is nonsingular [11],[23]. This concludes the proof that $\phi(z)$ can be uniquely parametrized by the set of covariances

$$\theta = [r_0, r_1, \ldots, r_{na+nc}] \; . \tag{2.8}$$

Another parametrization of $\phi(z)$ was considered by Walker [11] and Cadzow [12]. Walker parametrized $\phi(z)$ by

$$\theta = [r_0, r_1, \ldots, r_{nc}; a_1, \ldots, a_{na}] \; . \tag{2.9}$$

For $na < 2nc + 1$ it can be easily shown that $\phi(z)$ can be expressed as a function of the parameters in (2.9). For $na > 2nc + 1$, however, this is no longer that obvious. Walker [11] gave a formula expressing $\phi(z)$ as a function of (2.9), which appears to be in error. The simplicity of the parametrization of $\phi(z)$ via (2.8) was one of the reasons for preferring (2.8) to (2.9).

The parametrization of $\phi(z)$ used by Cadzow [12] (see also [13]) is shown in [33, Appendix D] to be a special case of (2.9), and is valid only for $nc > na$ (compared to the constraint $na < 2nc + 1$ mentioned above). Due to

this constraint it cannot be used for arbitrary ARMA processes.

Finally note that replacing $\{r_k, k=0,\ldots,na+nc\}$ in (2.4) - (2.6) by some (consistent) estimate will produce a (consistent) estimate of the spectral density, which is not guaranteed to be nonnegative on the unit circle. The same is true when using (2.9) to parametrize $\phi(z)$. This problem is discussed in more detail in [30][31], where a remedy is proposed.

3. A MAXIMUM LIKELIHOOD ESTIMATION PROBLEM AND ITS LARGE-SAMPLE SOLUTION

In the next section we will discuss an approximate ML method for estimating the covariances $\{r_k , k=0 ,\ldots, na+nc\}$ characterizing the ARMA process (2.1). In this section we present in a general setting the basic ideas behind that method. As was mentioned earlier, our approach follows that of Walker [11], who parametrized the ARMA process via (2.9). To estimate these parameters (more precisely, the parameters $\{r_1/r_0,\ldots,r_{nc}/r_0,a_1,\ldots,a_{na}\}$) he considered a more complicated approach than the one presented here. We formalized the basic ideas behind Walker's approach in [33, Appendix E]. We note that the approaches presented in this section and in [33, Appendix E] may be useful in deriving new estimators for other estimation problems besides the one treated here, see e.g. [25].

Let X be a random m-vector which is completely determined from the available N data samples. Let θ denote the $n\theta$-vector of unknown parameters to be estimated. Assume that for $N \to \infty$ the distribution of X is completely determined by θ . Furthermore, assume that

$$\sqrt{N}\,(X-\bar{X}) \xrightarrow[N\to\infty]{\text{dist}} \mathcal{N}(0,W) , \tag{3.1a}$$

where

$$\bar{X} = \begin{bmatrix} \theta \\ 0 \end{bmatrix}, \tag{3.1b}$$

and where the covariance matrix W (assumed to be nonsingular) may depend on θ . Finally, assume that an estimate $\hat{W}$ of W, which is such that

$|\hat{W}-W| = O(1/\sqrt{N})$[†] , can be calculated from the available data. Under these conditions we will derive a simple large sample approximate ML estimate of θ .

Since we consider the large-sample case, assumption (3.1a) is not too restrictive. Many statistics have an asymptotically Gaussian distribution according to various central limit theorems. The choice of X so as to fulfil (3.1b) is the critical point in applying the approach of this section to a specific estimation problem.

The asymptotic log-likelihood function of X is given by

$$L(\theta) = -\frac{m}{2}\ln 2\pi - \frac{1}{2}\ln\det W - \frac{N}{2}(X-\bar{X})^T W^{-1}(X-\bar{X}) \quad . \tag{3.2}$$

The ML estimate of θ obtained from a "sample" X drawn from the asymptotic distribution (3.1) is, therefore, the solution of the following equation

$$\frac{\partial L(\theta)}{\partial\theta} = -\frac{1}{2}\frac{\partial}{\partial\theta}[\ln\det W] + N[I_{n\theta},\ 0]W^{-1}(X-\bar{X})$$
$$-\frac{N}{2}\begin{bmatrix}(X-\bar{X})^T\dfrac{\partial W^{-1}}{\partial\theta_1}(X-\bar{X})\\ \vdots \\ (X-\bar{X})^T\dfrac{\partial W^{-1}}{\partial\theta_{n\theta}}(X-\bar{X})\end{bmatrix} = 0\ , \tag{3.3}$$

where θ_i is the i-th component of θ , and where $I_{n\theta}$ denotes the $n\theta \times n\theta$ unity matrix. Let us assume that (3.3) has a solution with respect to θ , say $\bar{\theta}$. Under certain regularity conditions the ML estimate is consistent and $|\bar{\theta} - \theta| = O(1/\sqrt{N})$. (We denote both the true and the unknown

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†We will use throughout the paper the notation $O(\varepsilon)$ to denote a random variable with standard deviation $K\varepsilon$, where ε is sufficiently small and where K is a (finite) constant independent of ε . An estimate $\hat{W}$ satisfying $|\hat{W}-W| = O(1/\sqrt{N})$ is sometimes called "root N consistent."

parameter vectors by the same symbol θ). Determination of $\bar{\theta}$ will in general be a highly intractable problem. In the following we derive an approximation of order 1/N of $\bar{\theta}$. For simplicity we assume that $m < \infty$. However, similar results hold if $m \to \infty$ and $m/N \to 0$ (as $m,N \to \infty$) at an "appropriate rate." The rate at which m should tend to infinity is not easy to determine, and will be problem dependent [23]-[24][37].

For N large enough it follows from (3.1) that

$$X - \begin{bmatrix} \bar{\theta} \\ 0 \end{bmatrix} = \left(X - \begin{bmatrix} \theta \\ 0 \end{bmatrix} \right) + \begin{bmatrix} \theta - \bar{\theta} \\ 0 \end{bmatrix} = O(1/\sqrt{N}) \; .$$

Thus we can rewrite (3.3) as,

$$\frac{1}{N} \frac{\partial L(\theta)}{\partial \theta} = [I_{n\theta} \quad 0] W^{-1} \; X - \begin{bmatrix} \bar{\theta} \\ 0 \end{bmatrix} + O(1/N) = 0 \; . \qquad (3.4)$$

Next we partition W and X as

$$W = \begin{array}{c} \begin{bmatrix} W_{11} & W_{12} \\ W_{12}^T & W_{22} \end{bmatrix} \begin{array}{l} \}n\theta \\ \\ \end{array} \\ \underbrace{\hphantom{W_{11}}}_{n\theta} \end{array} , \qquad (3.5)$$

$$X = \begin{bmatrix} x \\ z \end{bmatrix} \begin{array}{l} \}n\theta \\ \\ \end{array} .$$

A standard result on the inverse of partitioned matrices gives

$$W^{-1} = \begin{bmatrix} 0 \\ I \end{bmatrix} W_{22}^{-1} [0 \quad I] + \begin{bmatrix} -I \\ W_{22}^{-1} \; W_{12}^T \end{bmatrix} (W_{11} - W_{12} W_{22}^{-1} W_{12}^T)^{-1} [-I, \; W_{12} W_{22}^{-1}] \; . \qquad (3.6)$$

It follows from (3.4)-(3.6) that an asymptotic approximation of order 1/N of $\bar{\theta}$ is given by

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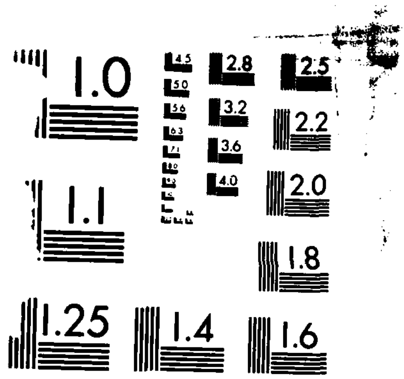

MICROCOPY RESOLUTION TEST CHART
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$$\underline{\theta} = x - W_{12}W_{22}^{-1}z \; . \tag{3.7}$$

Now $\underline{\theta}$ is not directly implementable since W_{12} and W_{22} will generally depend on θ . However, since $z = O(1/\sqrt{N})$, see (3.1), we can replace W_{ij} in (3.7) by their consistent estimates $\hat{W}_{ij}$ without affecting the order of the approximation. We can summarize the discussion above by the following lemma.

<u>Lemma 3.1</u>

Consider $\hat{\theta}$ given by

$$\hat{\theta} = x - \hat{W}_{12}\hat{W}_{22}^{-1}z \tag{3.8}$$

where x, $\hat{W}_{12}$, $\hat{W}_{22}$ and z are defined by (3.5). Then $\hat{\theta}$ is a simple large-sample approximate (of order 1/N) solution of (3.3).

Since $|\hat{\theta}-\bar{\theta}| = O(1/N)$, $\hat{\theta}$ has for $N \to \infty$ the same distribution as the ML estimate $\bar{\theta}$. In view of the asymptotic (for $m \to \infty$) efficiency of the ML estimate θ we expect that under certain regularity conditions the covariance matrix of the distribution of $\hat{\theta}$ will tend to the Cramér-Rao lower bound as $m \to \infty$. However this is only a conjecture. To prove it in specific cases is a challenging problem; see section 6 for the analysis of a particular case.

In the following we establish some general accuracy properties of $\hat{\theta}$. It follows from (3.1), (3.7) and (3.8) that

$$\sqrt{N}\,(\hat{\theta}-\theta) \xrightarrow[N \to \infty]{\text{dist}} \mathcal{N}(0,\, P_m) \; , \tag{3.9a}$$

where

$$P_m = W_{11}-W_{12}W_{22}^{-1}\, W_{12}^T \quad , \tag{3.9b}$$

and where, for the convenience of the discussion, we stress by notation the dependence of the covariance matrix (3.9b) on m.

The estimation error $(\hat{\theta}-\theta)$ can be interpreted as being the residuals of

the asymptotic regression of $x-\theta$ on z. Consider the following regression problem: determine $\hat{M}$ such that

$$Q(M) \triangleq NE\{[(x-\theta)-Mz][(x-\theta)-Mz]^T\} \geq Q(\hat{M}) , \tag{3.10}$$

for any $n\theta \times (m-n\theta)$ matrix M. Since we have

$$Q(M) = (M-W_{12}W_{22}^{-1})W_{22}(M-W_{12}W_{22}^{-1})^T + (W_{11}-W_{12}W_{22}^{-1}W_{12}^T) , \tag{3.11}$$

and since $W_{11}-W_{12}W_{22}^{-1}W_{12}^T$ and W_{22} are positive definite matrices, it follows that

$$\hat{M} = W_{12}W_{22}^{-1} , \tag{3.12}$$

and

$$Q(\hat{M}) = P_m , \tag{3.13}$$

According to the interpretation above we expect that the accuracy will increase when m increases since the number of "degrees of freedom" in the regression problem (3.10) increases with m. It can in fact be shown by straightforward algebraic calculations that

$$P_m \geq P_{\bar{m}} , \qquad \text{for } \bar{m} \geq m . \tag{3.14}$$

We state this result in the following theorem.

Theorem 3.1. Consider the matrices P_m and $P_{\bar{m}}$ defined by (3.9b) under the assumption that W_{22} is invertible. Assume that

$$z = [I_{m-n\theta} , 0]\bar{z} , \tag{3.15}$$

where z corresponds to P_m and $\bar{z}$ to $P_{\bar{m}}$. Then the order relation (3.14) holds true.

Proof: The nested structure (3.15) induces a similar structure on the matrices W_{12} and W_{22}, say

$$\overline{W}_{12} = [W_{12}, S_1] ,$$

and

$$\overline{W}_{22} = \begin{bmatrix} W_{22} & S_2 \\ S_2^T & S_3 \end{bmatrix} .$$

Thus

$$P_{\underline{m}} = W_{11} - \overline{W}_{12}\,\overline{W}_{22}^{-1}\,\overline{W}_{12}^T = W_{11} - [W_{12}, S_1] \cdot$$

$$\cdot \left\{ \begin{bmatrix} I \\ 0 \end{bmatrix} W_{22}^{-1}[I, 0] + \begin{bmatrix} W_{22}^{-1}S_2 \\ -I \end{bmatrix} S[S_2^T W_{22}^{-1} \quad -I] \right\} \begin{bmatrix} W_{12}^T \\ S_1^T \end{bmatrix}$$

$$= P_m - (S_1 - W_{12}\,W_{22}^{-1}\,S_2)S(S_1 - W_{12}\,W_{22}^{-1}\,S_2)^T , \tag{3.16}$$

where

$$S = (S_3 - S_2^T W_{22}^{-1}\,S_2)^{-1} .$$

Since $\overline{W}_{22}$ is positive definite by assumption, S must also be positive definite and the assertion of the theorem follows. We note from (3.16) that the equality

$$P_{\underline{m}} = P_m , \tag{3.17}$$

is equivalent to

$$S_1 = W_{12}\,W_{22}^{-1}\,S_2 . \tag{3.18}$$

Since $P_m > P_{m+1} > 0$ for all m, it follows that the sequence of matrices $\{P_m\}$ will have a limit when $m \to \infty$, which we denote by P_∞. According to the interpretation (3.13) of P_m the "rate of convergence" of P_m to P_∞ will be faster than that of the covariance matrix corresponding to any other

estimator of θ of the form x + Mz, for some matrix M. Also, according to the discussion following (3.8) we expect that

$$P_{\infty} = P_{CR} , \tag{3.19}$$

where P_{CR} denotes the asymptotic Cramér-Rao lower bound for consistent estimators of θ . This conjecture is analyzed for the specific case of the ARMA estimation problem in section 6.

Finally, note that the (consistent) estimate $\hat{\theta}$, (3.8), could be introduced independently of the ML interpretation. For example, it could be introduced using the (asymptotic) regression interpretation (3.10)-(3.13). The accuracy properties proven above ((3.9),(3.13) and (3.14)) do not depend on the ML interpretation of $\hat{\theta}$. The property (3.19) becomes, however, apparent only in relation to such an interpretation. Yet this property has to be proven in each particular case being considered. The application of the maximum likelihood principle in this section is non-standard. The likelihood function used here is valid only for $N \to \infty$. Moreover, it is not known whether the likelihood function is valid for $m \to \infty$. Thus, we can not rely on the standard properties of the ML estimate to prove (3.19). Note that for the specific ARMA problem considered in the next section we show that (3.19) holds for Gaussian data, but not necessarily for other distributions. In view of the discussion above, this should not be viewed as a contradiction to the ML-based interpretation of $\hat{\theta}$.

4. LARGE-SAMPLE MAXIMUM LIKELIHOOD ARMA SPECTRAL ESTIMATION

In this section we consider the specific problem of estimating the spectral density of an ARMA process (2.1). This problem reduces to estimating the covariance parameters $\theta \triangleq [r_0, \ldots, r_{na+nc}]$ of the ARMA process, see (2.4) - (2.6). We will use the approximate ML approach of the previous section to estimate θ from a sample $\{y(1),\ldots,y(N)\}$. We define the unbiased sample covariances

$$\begin{aligned} \tilde{r}_k &= \frac{1}{N-k} \sum_{t=1}^{N-k} y(t)\, y(t+k) , \qquad k = 0,1,2,\ldots, \\ \tilde{r}_{-k} &= \tilde{r}_k . \end{aligned} \tag{4.1}$$

Next we introduce a consistent estimate of the AR parameters $\{a_i\}$ obtained by the least-squares solution of the following overdetermined Yule-Walker system of equations

$$\begin{bmatrix} \tilde{r}_{nc} & \cdots & \tilde{r}_{nc+1-na} \\ \vdots & & \vdots \\ \tilde{r}_{K-1} & \cdots & \tilde{r}_{K-na} \end{bmatrix} \tilde{a} = - \begin{bmatrix} \tilde{r}_{nc+1} \\ \vdots \\ \tilde{r}_{K} \end{bmatrix}, \quad K > na+nc, \tag{4.2}$$

where $\tilde{a} = [\tilde{a}_1 \ \ldots \ \tilde{a}_{na}]^T$. That $\tilde{a}$ given by (4.2) is a consistent estimate of $a = [a_1, \ldots, a_{na}]^T$ follows readily from (2.4) and from the convergence of the sample covariances to the theoretical covariances $\{r_k\}$ [14],[23]. Note that the sample covariance matrix in (4.2) has full rank, at least for sufficiently large N [23]. It may be advisable to take K in (4.2) to be much larger than na+nc in order to improve the accuracy of $\tilde{a}$ [7]. It is not generally true that increasing K improves the accuracy of $\tilde{a}$ [23]. However, extensive simulations [7],[12] have shown that this is in general the case when the sequence $\{|r_k|\}$ is decreasing slowly.

Next we define the statistic X which will constitute the "data" for our ML estimation problem:

$$X = \begin{bmatrix} x \\ z \end{bmatrix}, \quad x = \begin{bmatrix} x_0 \\ \vdots \\ x_{na+nc} \end{bmatrix}, \quad z = \begin{bmatrix} z_1 \\ \vdots \\ z_{m-na-nc-1} \end{bmatrix}, \tag{4.3a}$$

where

$$x_i = \tilde{r}_i \ , \qquad i=0,\ldots,na+nc \ ,$$

$$z_k = \sum_{i=0}^{na} \sum_{j=0}^{na} \tilde{a}_i \ \tilde{a}_j \ \tilde{r}_{nc+na+k-i-j} \ , \quad k=1,\ldots,m-na-nc-1 \ . \tag{4.3b}$$

We assume that $m > na + nc + 1$. The specific form above of the vector z leads to a relatively simple expression for the covariance matrix of the asymptotic distribution of X (see below). Other choices of z are possible but we believe

(4.3b) is the most convenient choice. This choice of z was introduced by Walker [11].

It is shown in Appendix A that X in (4.3) is asymptotically normally distributed, i.e.,

$$\sqrt{N}\,(X-\bar{X}) \xrightarrow[N\to\infty]{\text{dist}} \mathcal{N}(0,\ W)\ , \tag{4.4a}$$

where

$$\bar{X} = \begin{bmatrix} r_0 \\ \vdots \\ r_{na+nc} \\ \hline 0 \end{bmatrix} \triangleq \begin{bmatrix} \theta \\ \\ \hline 0 \end{bmatrix}\ , \tag{4.4b}$$

and

$$W = \begin{bmatrix} W_{11} & W_{12} \\ W_{12}^T & W_{22} \end{bmatrix} \begin{array}{l} \} \ n\theta \\ \\ \end{array} \quad , \ n\theta \triangleq na + nc + 1\ .\ ,$$

with the first column of W of width $n\theta$.

$$[W_{11}]_{ij} = E\{[v(t-i) + v(t+i)]v(t-j)\} \quad i,j = 0,\dots,n\theta-1\ , \tag{4.4c}$$

$$v(t) = \lambda\,\frac{C^2(q^{-1})}{A^2(q^{-1})}\,e(t),$$

$$\begin{aligned}[W_{22}]_{ij} &= E\{A^2(q^{-1})v(t-i)A^2(q^{-1})v(t-j)\} \\ &= \text{the coefficient of } z^{i-j} \text{ in } \Big[\sum_{k=-nc}^{nc} b_k\, z^{-k}\Big]^2, \quad i,j=1,\dots,m-n\theta\end{aligned} \tag{4.4d}$$

$$\begin{aligned}[W_{12}]_{jk} &= E\{A^2(q^{-1})[v(t+na+nc-j) + v(t+na+nc+j)]v(t-k)\} \\ &= \alpha_{k+j} + \alpha_{k-j}\ , \qquad j=0,\dots,n\theta-1, \qquad k=1,\dots,m-n\theta\ ,\end{aligned} \tag{4.4e}$$

$$\alpha_s = \text{the coefficient of } z^s \text{ in the long division of } z^{-(nc+na)}\,\frac{\Big[\sum_{k=-nc}^{nc} b_k\, z^{-k}\Big]^2}{A^2(z^{-1})}\,.$$

It is not difficult to see that

$$[W_{12}]_{jk} = 0 \qquad \text{for} \qquad k > nc-na+j \ . \tag{4.4f}$$

This implies that $[W_{12}]_{jk} = 0$ for $k > 2nc$. Note also that W_{22} is a banded Toeplitz matrix with the band width equal to $2nc + 1$.

In (4.4d) and (4.4e) we have indicated simple ways for evaluating the covariance matrices W_{12} and W_{22}. Note that only these two matrices are of interest in calculating the estimate, cf. (3.7). The matrices W_{12} and W_{22} depend only on $\{a_i, i=1, \ldots, na\}$ and $\{r_k, k=0,\ldots,na+nc\}$ .Thus, consistent estimates of W_{12} and W_{22} can be obtained by using in (4.4) the consistent estimates of $\{a_i\}$ and $\{r_k\}$ given by (4.1) and (4.2).

It follows from the discussion above that X (4.3), satisfies the basic conditions used to develop the approximate ML approach of section 3. Thus, a large-sample approximation of the ML estimate of θ is given by (lemma 3.1),

$$\hat{\theta} = x - \hat{W}_{12}\,\hat{W}_{22}^{-1} z \tag{4.5}$$

The W_{22} matrix is positive definite for any value of m (finite or infinite) [27][33]. More precisely, it can be shown that

$$\lambda_{min}(W_{22}) \geq \lambda^4 \inf_{\omega} |C(e^{i\omega})|^4 \tag{4.6}$$

$$\lambda_{max}(W_{22}) \leq \lambda^4 \sup_{\omega} |C(e^{i\omega})|^4 \tag{4.7}$$

where $\lambda_{min}(W_{22})$ and $\lambda_{max}(W_{22})$ are the smallest and largest eigenvalues of the matrix W_{22} , respectively. The equalities in both (4.6) and (4.7) hold in the limit as $m \to \infty$ [33, Appendix F]. Due to assumption A1 we have $W_{22} > 0$ for all m. Note, however, that if C(z) has zeroes near the unit circle then the condition number of W_{22} will be large for large values of m. A similar situation will occur for $\hat{W}_{22}$. Some numerical problems may arise in such a case in the implementation of the estimator defined by (4.5).

The algorithm for determining a large-sample ML estimate of $\phi(z)$ based on (4.5) can be summarized as follows:

Step 1. Compute the sample covariances $\{\tilde{r}_k\}$ (4.1), and the initial estimate $\tilde{a}$ (4.2).

Step 2 Use $\{\tilde{r}_k\}$, $\tilde{a}$ in (2.5) to obtain initial estimates $\{\tilde{b}_k\}$ and insert them in (4.3) and (4.4) to compute x, z, $\hat{W}_{12}$ and $\hat{W}_{22}$. Compute improved estimates $\{\hat{r}_k, k=0, \ldots, na+nc\}$ of the covariances by using (4.5).

Step 3. Use $\{\hat{r}_k, k=0, \ldots, na+nc\}$ in (2.4) with $k=nc+1,\ldots,nc+na$, to obtain an improved estimate $\hat{a}$ of the AR parameters. Then use $\hat{a}$ and $\{\hat{r}_k, k=0, \ldots, na+nc\}$ in (2.6) to obtain the estimate $\hat{\phi}(z)$ of the ARMA spectral density.

The calculations in steps 2 and 3 of the above algorithm can be repeated using the improved estimates $\{\hat{r}_i\}$ and $\{\hat{a}_i\}$ obtained in step 3. For large N this will have only a slight effect on the estimates. However, in the small and medium sample cases the iteration of steps 2 and 3 may have a beneficial effect on estimation accuracy.

The computational aspects related to the algorithm above are discussed in detail in [32]. Here we note only that the facts that W_{22} is a banded positive definite matrix and that W_{12} has few non-zero elements can be exploited to get a computationally efficient algorithm (requiring proportional to m arithmetic operations) for implementing steps 2-3.

Some general accuracy properties of the estimates of the type given in equation (4.5) have been derived in section 3. Analogous properties clearly hold for the estimates $\hat{a}$ and $\hat{\phi}(z)$ obtained by the algorithm above. A more detailed accuracy analysis of $\hat{\theta}$ and $\hat{a}$ will be presented later.

We conclude this section by noting that Walker [11], who used a somewhat more cumbersome ML approach [33, Appendix E] arrived at estimates of the correlations $\{r_k/r_0, k=1, \ldots, nc\}$ and of the AR parameters $\{a_i\}$ that are similar to ours. Since Walker considered the estimation of $\{r_k/r_0\}$ instead of $\{r_k\}$, our estimates and those of Walker cannot be easily compared.

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We conclude this section by noting that Walker [11], who used a somewhat more cumbersome ML approach [33, Appendix E] arrived at estimates of the correlations $\{r_k/r_0, k=1, \ldots, nc\}$ and of the AR parameters $\{a_i\}$ that are similar to ours. Since Walker considered the estimation of $\{r_k/r_0\}$ instead of $\{r_k\}$, our estimates and those of Walker cannot be easily compared.

5. ASYMPTOTIC ACCURACY PROPERTIES

In this section we derive explicit expressions for the asymptotic covariance matrices of $\hat{\theta}$ and $\hat{a}$. The asymptotic properties of $\hat{\phi}(z)$ can be analyzed similarly (pointwise).

It follows from the general analysis in section 3 that as $N \to \infty$ the covariance matrix of the normalized estimation error $\sqrt{N}(\hat{\theta} - \theta)$, is given by

$$P_m^\theta = W_{11} - W_{12} W_{22}^{-1} W_{12}^T , \tag{5.1}$$

where the matrices W_{ij} are defined by (4.4). Furthermore, according to Theorem 3.1 we have

$$P_m^\theta > P_{\bar{m}}^\theta , \qquad \text{for } \bar{m} > m . \tag{5.2}$$

A consequence of (5.2) is that the sequence of positive definite matrices $\{P_m^\theta\}$ has a limit when $m \to \infty$, which we denote P_∞^θ. An explicit expression for P_∞^θ is given in the following lemma.

<u>Lemma 5.1</u> Consider the covariance matrix P_m^θ defined by (5.1), (4.4). Then

$$P_\infty^\theta = W_{11} - \Omega\,\Omega^T \tag{5.3a}$$

where Ω is a $n\theta \times 2nc$ matrix whose (i,j)-element is given by

$$\Omega_{ij} = E\{C^2(q^{-1})[e(t-i) + e(t+i)] \frac{1}{A^2(q^{-1})} e(t-nc-na-j)\} , \tag{5.3b}$$

$$i = 0,\ldots,n\theta-1, \; j = 1,\ldots,2nc .$$

<u>Proof:</u>

Let

$$\sum_{j=0}^{\infty} h_j z^j \triangleq \frac{1}{C^2(z)} , \qquad (h_0 = 1) . \tag{5.4}$$

It is shown in [10] that for $m \to \infty$ the (i,j)-element of W_{22}^{-1} is given by

$$\lim_{m\to\infty} [W_{22}^{-1}]_{ij} = \frac{1}{\lambda^4} \sum_{r=1}^{\min(i,j)} h_{i-r}\, h_{j-r} \,, \qquad i,j \geq 1 \,. \tag{5.5}$$

Thus, we can write

$$\lim_{m\to\infty} W_{22}^{-1} = \frac{1}{\lambda^4} U^T U \,, \tag{5.6a}$$

where U is the following infinite-dimensional matrix

$$U = \begin{bmatrix} h_0 & h_1 & h_2 & h_3 & \cdots \\ & h_0 & h_1 & h_2 & \cdots \\ & & h_0 & h_1 & \cdots \\ 0 & & & \ddots & \ddots \end{bmatrix} . \tag{5.6b}$$

Since

$$U \begin{bmatrix} v(t-1) \\ v(t-2) \\ \vdots \end{bmatrix} = \frac{1}{C^2(q^{-1})} \begin{bmatrix} v(t-1) \\ v(t-2) \\ \vdots \end{bmatrix} = \frac{\lambda}{A^2(q^{-1})} \begin{bmatrix} e(t-1) \\ e(t-2) \\ \vdots \end{bmatrix} ,$$

the assertion of the lemma readily follows from (4.4e) and (5.1). Note that the expression in (5.3b) becomes zero if $j > 2nc$.

In practice m cannot be too large. The computational burden increases in proportion to m. Also, m must be only a fraction of the sample size N for statistical "stability". The rate of convergence of P_m^θ to its limiting lower bound P_∞^θ given by (5.3) is, therefore, of interest. Due to the particular structure of W_{12} (4.4f), the rate of convergence of P_m^θ to P_∞^θ is determined essentially by the rate at which the left-top 2ncx2nc-block of W_{22}^{-1} converges as $m \to \infty$. The rate of convergence of the entries in that block depends strongly on the location of the zeros of C(z), see (5.4)-(5.6). The closer these zeros are to the unit circle, the slower is the convergence rate. The parameters $\{a_i\}$ have a much smaller influence on the convergence rate of P_m^θ , via the elements of the non-zero block of W_{12} .

We conclude from the discussion above that in the case $C(z)$ has zeros well inside the unit circle we can get resonably close to the lower bound P_∞^θ for relatively small values of m. If $C(z)$ has zeros close to the unit circle we may need to consider much larger values of m. This will be possible only if we have a long sample at hand; otherwise we cannot attain the maximum accuracy corresponding to P_∞^θ . Recall also that for m large and $C(z)$ with zeros close to the unit circle, the $\hat{W}_{22}$ matrix is likely to be ill-conditioned.

Next we turn to the calculation of the asymptotic covariance matrix of $\hat{a}$. We introduce the vector

$$\hat{r} = [\hat{r}_{nc+1},\dots,\hat{r}_{nc+na}]^T, \tag{5.7}$$

and the matrix $\hat{R}$ as defined by (2.7) with $\{r_i\}$ replaced by $\{\hat{r}_i\}$. The estimate $\hat{a}$ can be written as a function of $\hat{\theta}$ as follows

$$\hat{a} = -\hat{R}^{-1}\hat{r} . \tag{5.8}$$

We can now state the following result.

Lemma 5.2: Consider the estimate $\hat{a}$, (5.8), where $\{\hat{r}_i\}$ are given by (4.5). Let P_m^a denote the asymptotic covariance matrix of $\sqrt{N}(\hat{a}-a)$. Then

$$P_m^a = R^{-1}(Q_{11} - Q_{12}\, W_{22}^{-1}\, Q_{12}^T)R^{-T} , \tag{5.9}$$

where W_{22} is given by (4.4d), R by (2.7), and

$$[Q_{11}]_{ij} = E\{A(q^{-1})v(t-i)\ A(q^{-1})v(t-j)\} , \quad i,j=1,\dots,na , \tag{5.10}$$

$$[Q_{12}]_{ij} = E\{A^2(q^{-1})\ v(t-i)\ A(q^{-1})\ v(t-na-j)\} , \quad \begin{array}{l} i=1,\dots,na , \\ j=1,\dots,m-n\theta . \end{array}$$

Furthermore,

$$P_m^a > P_{\bar{m}}^a , \quad \text{for} \quad \bar{m} > m . \tag{5.11}$$

Finally, the limit covariance matrix $P_\infty^a = \lim_{m \to \infty} P_m^a$ exists and is given by

$$P_\infty^a = R^{-1}(Q_{11} - \Gamma\ \Gamma^T)R^{-T} ,$$

where

$$[\Gamma]_{ij} = E\{C^2(q^{-1})\ e(t-i)\ \frac{1}{A(q^{-1})}\ e(t-na-j)\} , \quad \begin{array}{l} i=1,\ldots\ldots, na , \\ j=1,\ldots\ldots, 2nc . \end{array} \tag{5.12}$$

Proof:

See Appendix B.

As stated earlier, our estimate $\hat{a}$ may differ from Walker's estimate for finite samples. A careful comparison of P_m^a with the expression given by Walker for the covariance matrix of the estimate in [11] shows that they are identical. Thus, the two estimates have the same asymptotic accuracy.

Note that the first term in (5.9), $R^{-1}Q_{11}R^{-T}$ is the covariance matrix corresponding to the standard Yule-Walker estimate of a (i.e., $\tilde{a}$ obtained from (4.2) for K = na+nc), see [23]. Thus, the second term in (5.9) shows the improvement in asymptotic accuracy that results by using $\{\hat{r}_k\}$ instead of $\{\tilde{r}_k\}$ in the basic Yule-Walker equations.

In the next section we compare the asymptotic accuracy of our estimates to the Cramér-Rao lower bound. In the course of the analysis we also obtain an interesting result relating the covariance matrix of $\hat{a}$, to the covariance matrix of the optimal Yule-Walker estimate recently proposed in [23][24].

6. COMPARISON WITH THE CRAMÉR-RAO LOWER BOUND

The following conjecture was introduced previously:

$$P_\infty^\theta = P_{CR}^\theta , \tag{6.1}$$

where P_{∞}^{θ} is given by (5.3) and P_{CR}^{θ} is the CRLB for the covariance matrix of any consistent estimator of θ. In the sequel we present a proof of (6.1) for the general ARMA case, using some results presented in [34], [35]. The asymptotic (for N, $m \to \infty$) efficiency of the approximate ML estimators of the type considered here was conjectured by Walker [11] and later by others [16][28], but no proof was provided, except in some special cases [9],[37]. Explicit expressions for P_{CR}^{θ} are known [34][36]. However, a direct algebraic proof of the equivalence between P_{∞}^{θ} in (5.3) and P_{CR}^{θ} appears to be difficult. Instead we consider the following result introduced in [35].

Theorem 6.1

Let $\hat{\hat{\theta}}$ be the following estimate of the ARMA parameter vector θ (2.8),

$$\hat{\hat{\theta}} = \underset{\theta}{\operatorname{argmin}}\, V(\theta), \tag{6.2a}$$

$$V(\theta) = \eta^T \Sigma^{-1} \eta \;, \tag{6.2b}$$

where

$$\eta = [\tilde{r}_0 - r_0, \; \ldots, \; \tilde{r}_{m-1} - r_{m-1}]^T, \tag{6.2c}$$

$$\Sigma \triangleq \lim_{N\to\infty} N \operatorname{cov}\{\eta\} \;. \tag{6.2d}$$

and where $m > na+nc+1$. Define

$$\bar{P}_m \triangleq \lim_{N\to\infty} N \operatorname{cov}\{\hat{\hat{\theta}}\} \;. \tag{6.3}$$

Then, under the Gaussian hypothesis

$$\lim_{m\to\infty} \bar{P}_m = P_{CR}^{\theta} \;. \tag{6.4}$$

Proof: See [35].

It is also shown in [35] that $\bar{P}_m$ is an asymptotic lower bound on the covariance matrix of any estimator based on the m sample covariances

$\{\tilde{r}_0, \ldots, \tilde{r}_{m-1}\}$.

Next we state and prove the following lemma.

Lemma 6.1:

The estimate $\hat{\theta}$ as defined by (4.5) and the estimate $\hat{\hat{\theta}}$ defined by (6.2a) are asymptotically equivalent.

Proof: Let

$$\bar{z} = [\bar{z}_1 \; \ldots \; \bar{z}_{m-na-nc-1}]^T \tag{6.5a}$$

where $\{\bar{z}_k\}$ are defined by (A.16). Also define

$$\tilde{X} = \begin{bmatrix} x \\ \bar{z} \end{bmatrix}$$

where x is given by (4.3a). Then

$$\tilde{X} - \bar{X} = \underbrace{\begin{bmatrix} 1 & & 0 \\ & \ddots & \\ * & & 1 \end{bmatrix}}_{V} \eta \tag{6.5c}$$

where $\bar{X}$, η are defined in (4.4b) and (6.2c), and * denotes entries whose values are not important for this proof. Since the matrix V is nonsingular and non-random it follows that

$$V(\theta) \triangleq \eta^T \Sigma^{-1} \eta = (\tilde{X} - \bar{X})^T W^{-1} (\tilde{X} - \bar{X}) \tag{6.6}$$

where $W = \lim_{N \to \infty} N \operatorname{cov}(\tilde{X}-\bar{X})$ is defined in (4.4) (see also Appendix A). Note that we denote both the true and the unknown parameter vectors by the true symbol θ . The equality (6.6) holds for all admissible values of θ . Thus we have

$$\frac{\partial V(\theta)}{\partial\theta} = 2\,\frac{\partial\tilde{X}^T}{\partial\theta}\,W^{-1}(\tilde{X}-\bar{X}) - 2[I\ 0]\,W^{-1}(\tilde{X}-\bar{X}) + \begin{bmatrix} (\tilde{X}-\bar{X})^T\,\frac{\partial W^{-1}}{\partial\theta_1}\,(\tilde{X}-\bar{X}) \\ \vdots \\ (\tilde{X}-\bar{X})^T\,\frac{\partial W^{-1}}{\partial\theta_{n\theta}}\,(\tilde{X}-\bar{X}) \end{bmatrix} \tag{6.7}$$

Next note that $\hat{\hat{\theta}}$ is a root N consistent estimate: $|\hat{\hat{\theta}}-\theta| = o(1/\sqrt{N})$ (see (6.3)). Therefore

$$(\tilde{X}-\bar{X})\Big|_{\hat{\hat{\theta}}} = o(1/\sqrt{N}) \tag{6.8}$$

Furthermore, it follows exactly as in (A.1), (A.2) that

$$\frac{\partial\tilde{X}^T}{\partial\theta}\Big|_{\hat{\hat{\theta}}} = [0\ \frac{\partial\bar{Z}^T}{\partial\theta}\Big|_{\hat{\hat{\theta}}}\ ] = o(1/\sqrt{N}) \tag{6.9}$$

which implies that

$$\tilde{X}\Big|_{\hat{\hat{\theta}}} = \tilde{X}\Big|_{\bar{\theta}} + \frac{\partial\tilde{X}}{\partial\theta}\Big|_{\hat{\hat{\theta}}}(\hat{\hat{\theta}} - \bar{\theta}) + o(1/N) = X + o(\tfrac{1}{N}) \tag{6.10}$$

where $\bar{\theta}$ corresponds to $\{\bar{a}_i\}$ (note that $X \triangleq \tilde{X}|_{\bar{\theta}}$, see (4.3)). From (6.7)-(6.9) it follows that $\hat{\hat{\theta}}$ satisfies the following equation

$$[I\ 0]W^{-1}(X - [\begin{smallmatrix}\hat{\hat{\theta}}\\ 0\end{smallmatrix}]) + o(\tfrac{1}{N}) = 0 \tag{6.11}$$

Since $\hat{\theta}$ is the approximate (of order 1/N) solution of an equation with identical dominant term (see (3.4)), we conclude that $\hat{\theta} - \hat{\hat{\theta}} = o(1/N)$, and the proof is finished.

From Lemma 6.1 we conclude that $P_m^{\theta} = \bar{P}_m$. Thus, $\hat{\theta}$ is a minimum variance estimator in the class of estimators based on m sample covariances $\{\tilde{r}_0,\ldots,\tilde{r}_{m-1}\}$. Furthermore, from theorem 6.1 it follows that

$$\lim_{m\to\infty} P_m^{\theta} = P_{CR}^{\theta} \tag{6.12}$$

Thus, $\hat{\theta}$ is an asymptotically (for $N, m \to \infty$) efficient estimator. An

immediate consequence of this fact is that both $\hat{a}$ and $\hat{\phi}(z)$ (are asymptotically efficient estimators. Of less importance is the fact that the above results provide another way for introducing the estimator $\hat{\theta}$ (as a large-sample approximation of $\hat{\hat{\theta}}$).

In the remaining part of this section we show that P^a_m is equal to the asymptotic covariance of the optimal YW estimator of a, introduced in [23]. Since the optimal YW estimator is asymptotically efficient [23], this equality provides an alternative proof of the asymptotic efficiency of $\hat{a}$. The equivalence between the covariance matrices of these two estimators is also interesting in its own right.

Let us introduce the matrices R_k (k x na) and S_k(k x k), for $k > na$,

$$
\begin{aligned}
[R_k]_{ij} &= E\{y(t-nc-i)y(t-j)\} \qquad \begin{matrix} i=1,\dots,k , \\ j=1,\dots,na ; \end{matrix} \\
[S_k]_{ij} &= \lambda^2 E\{C(q^{-1})y(t-i)C(q^{-1})y(t-j)\} , \qquad i,j = 1,\dots,k ,
\end{aligned}
\tag{6.13}
$$

and $\bar{P}_K$ defined as

$$
\bar{P}_k = (R_k^T S_k^{-1} R_k)^{-1} . \tag{6.14}
$$

The inverse matrix in (6.14) exists for any $k > na$ [23].

The following result relating P^a_m to $\bar{P}_k$ (for a certain k) is essential in proving that $P^a_{CR} = P^a_\infty$.

<u>Theorem 6.2</u>: Consider the covariance matrices P^a_m and $\bar{P}_k$ defined by (5.9)-(5.10) and (6.13)-(6.14), respectively. Let $m > na + nc + 1$. Then

$$
P^a_m = \bar{P}_{m-nc-1} . \tag{6.15}
$$

<u>Proof:</u> See Appendix C.

Note that $\bar{P}_k$ $(k > na)$ is the asymptotic covariance matrix of the

optimally weighted overdetermined Yule-Walker estimator (OWOYWE) (or the asymptotically equivalent optimal IV estimate) recently introduced in [23]. Thus, $\hat{a}$ given by (5.8) and the OWOYWE of [23] based on m-nc-1 instruments have the same accuracy, as $N \to \infty$. These two estimates seem, in fact, to be asymptotically identical; however, in the finite-sample case they will in general have different values.

The reason for the usefulness of equality (6.15) is that the convergence of $\bar{P}_k$ as $k \to \infty$ was studied in detail in [23]. In particular it was shown there that under the Gaussian hypothesis

$$\bar{P}_k \to P^a_{CR} \quad , \qquad \text{as} \quad k \to \infty . \tag{6.16}$$

An explicit expression for P^a_{CR} was also given in [23]. The "rate of convergence" of $\bar{P}_k$ to P^a_{CR} was also studied in [23] by means of some numerical examples as well as analytical calculations. The results reported there on the convergence rate in (6.16) support the statements we already made in section 5: (i) the C-parameters have a much stronger influence on the convergence rate than the A-parameters; (ii) the convergence is slow when C(z) has zeros close to the unit circle.

7. CONCLUSIONS

We developed a technique for estimating the spectral parameters of an ARMA process from a set of sample covariances. The proposed algorithm provides consistent parameter estimates. Explicit expressions were derived for the asymptotic covariances of the parameter estimates. It was shown that the estimates of the ARMA parameters obtained by this technique are asymptotically efficient.

The computational requirements of the proposed technique are of the same order as those of the modified Yule-Walker estimator. A more detailed discussion of the computational and implementation aspects of this algorithm and a numerical performance evaluation will be presented in [32].

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APPENDIX A: PROOF OF EQUATION (4.4)

Let us consider the following Taylor series expansion of z_k, (4.3b) (viewed as a function of $\tilde{a}$), around a:

$$z_k(\tilde{a}) = \bar{z}_k + \sum_{s=1}^{na} \frac{\partial z_k(\tilde{a})}{\partial \tilde{a}_s}\bigg|_{\tilde{a}=a} \cdot (\tilde{a}_s - a_s) + O(1/N) , \tag{A.1a}$$

where

$$\bar{z}_k \triangleq \sum_{i=0}^{na} \sum_{j=0}^{na} a_i a_j \tilde{r}_{nc+na+k-i-j} , \qquad k > 1 , \tag{A.1b}$$

and

$$\frac{\partial z_k(\tilde{a})}{\partial \tilde{a}_s}\bigg|_{\tilde{a}=a} = 2 \sum_{j=0}^{na} a_j \tilde{r}_{(nc+na+k-s)-j} \qquad , k > 1 , \quad 1 < s < na . \tag{A.1c}$$

According to the Yule-Walker equations (2.4) the derivative (A.1c) is $O(1/\sqrt{N})$. It then follows from (A.1a) that

$$z_k(\tilde{a}) = \bar{z}_k + O(1/N) \quad . \tag{A.2}$$

Thus, the random variables z_k and $\bar{z}_k$ have the same asymptotic behavior, and in the following calculations we will consider $\bar{z}_k$ instead of z_k.

Under the assumptions imposed on the ARMA process (2.1) it is well known that for any finite k

$$\sqrt{N}\begin{bmatrix} \tilde{r}_0 - r_0 \\ \vdots \\ \tilde{r}_k - r_k \end{bmatrix} \xrightarrow[N \to \infty]{\text{dist}} \mathcal{N}(0, V) \quad , \tag{A.3a}$$

where

$$[V]_{ij} = \sum_{\tau=-\infty}^{\infty} (r_\tau r_{\tau+j-i} + r_\tau r_{\tau+i+j}) \quad , \tag{A.3b}$$

see [14],[15]. Since $\{x_i\}$, (4.3b), and $\bar{z}_k$, (A.1b), are linear combinations of $\{\tilde{r}_j\}$, the convergence in distribution of X, (4.3), to a Gaussian distribution follows from (A.3). It remains to verify the expression of the covariance matrix of the limiting distribution, given in (4.4). Note that formulae analogous to (4.4) have been given, without any proof, in [11].

<u>Proof of (4.4c)</u>

Let

$$\phi_k = \phi_{-k} \triangleq \sum_{s=-\infty}^{\infty} r_s r_{s+k} \quad . \tag{A.4}$$

Note from (A.3) that

$$[W_{11}]_{ij} = [V]_{ij} = \phi_{j-i} + \phi_{j+i} \quad . \tag{A.5}$$

Now,

$$\sum_{k=-\infty}^{\infty} \phi_k \; z^{-k} = \sum_{s=-\infty}^{\infty} r_s \sum_{k=-\infty}^{\infty} r_{s+k} z^{-k} =$$

$$= \left(\sum_{s=-\infty}^{\infty} r_s z^s \right) \left(\sum_{p=-\infty}^{\infty} r_p \; z^{-p} \right) = \phi^2(z) \; . \tag{A.6}$$

Thus, ϕ_k is equal to the covariance at lag k of the process

$$v(t) = \lambda \, \frac{C^2(q^{-1})}{A^2(q^{-1})} \, e(t) \; , \tag{A.7}$$

and the proof of (4.4c) is concluded.

Note that to compute the estimate (4.5) we do not need to consistently estimate W_{11}. However, a need for calculating W_{11} could arise if we want to compute the covariance matrix P_m^{θ}, (5.1), or its limit as $m \to \infty$, (5.3). To evaluate the entries of W_{11} we cannot proceed by "long division" as we did for W_{12} and W_{22}, see (4.4). The reason is that the coefficient ϕ_k of z^k in the infinite division of $\phi^2(z)$ cannot be computed without truncation errors. Instead, we can calculate ϕ_k as

$$\phi_k = \frac{\lambda^4}{2\pi i} \oint \frac{C^2(z) C^2(z^{-1})}{A^2(z) A^2(z^{-1})} \, z^k \, \frac{dz}{z} = \frac{1}{2\pi i} \oint \phi^2(z) z^k \, \frac{dz}{z} \quad , \tag{A.8}$$

by using an exact algorithm for evaluating complex integrals given in [26].

<u>Proof of (4.4d)</u>

We have that, cf. (A.2) and (A.5)

$$[W_{22}]_{ij} = \sum_{k=0}^{na} \sum_{p=0}^{na} \sum_{\ell=0}^{na} \sum_{s=0}^{na} a_k a_p a_\ell a_s \lim_{N\to\infty} NE\{(\tilde{r}_{na+nc+i-k-p} -$$

$$- r_{na+nc+i-k-p})(\tilde{r}_{na+nc+j-\ell-s} - r_{na+nc+j-\ell-s})\} =$$

$$= \sum_{k=0}^{na} \sum_{p=0}^{na} \sum_{\ell=0}^{na} \sum_{s=0}^{na} a_k a_p a_\ell a_s [\phi_{j-i+k+p-\ell-s} +$$

$$+ \phi_{2na+2nc+i+j-k-p-\ell-s}] \, . \tag{A.9}$$

Let us denote the two terms in (A.9) by T_1 and T_2. According to the interpretation (A.7) of ϕ_k we can write

$$\begin{aligned} T_1 &= E\{\sum_{k,p,\ell,s=0}^{na} a_k a_p a_\ell a_s v(t-i-\ell-s) \cdot v(t-j-k-p)\} = \\ &= E\{A^2(q^{-1})v(t-i)\; A^2(q^{-1})v(t-j)\} \, , \end{aligned} \tag{A.10}$$

and

$$T_2 = E\{\sum_{k,p,\ell,s=0}^{na} a_k a_p a_\ell a_s \; v(t-2na-2nc)\; v(t+i+j-k-p-\ell-s)\} =$$

$$= E\{v(t-2na-2nc) \cdot \lambda\; A^2(q^{-1})C^2(q^{-1})e(t+i+j)\} = 0, \quad \text{for } i,j > 1 \, . \tag{A.11}$$

We thus get

$$[W_{22}]_{ij} = E\{A^2(q^{-1})\; v(t-i)A^2(q^{-1})\; v(t-j)\} \, , \tag{A.12}$$

which is the expression given in (4.4d). To complete the proof of (4.4d) we note that $[W_{22}]_{kj}$ can also be written as

$$\begin{aligned} [W_{22}]_{kj} &= \frac{\lambda^4}{2\pi i} \oint C^2(z)\; C^2(z^{-1}) z^{k-j} \frac{dz}{z} = \\ &= \frac{1}{2\pi i} \oint [\sum_{s=-nc}^{nc} b_s \; z^{-s}]^2 \; z^{k-j} \frac{dz}{z} \, . \end{aligned} \tag{A.13}$$

Proof of (4.4e)

It follows from (A.2), (A.5) and (A.7) that

$$[W_{12}]_{ij} = \sum_{k=0}^{na} \sum_{p=0}^{na} a_k a_p \lim_{N\to\infty} NE\{(\tilde{r}_i - r_i)(\tilde{r}_{na+nc+j-k-p} -$$

$$-r_{na+nc+j-k-p})\} = \sum_{k=0}^{na} \sum_{p=0}^{na} a_k a_p [\phi_{na+nc+j-i-k-p} + \phi_{na+nc+j+i-k-p}]$$

$$= E\{\sum_{k,p=0}^{na} a_k a_p [v(t-i-k-p)v(t-na-nc-j)$$

$$+ v(t+i-k-p)v(t-na-nc-j)]\} =$$

$$= E\{A^2(q^{-1})[v(t-i)+v(t+i)]v(t-na-nc-j)\} ,$$

which can also be written as

$$[W_{12}]_{kj} = \frac{\lambda^4}{2\pi i} \oint \frac{C^2(z)C^2(z^{-1})}{A^2(z^{-1})} z^{-(na+nc)}[z^{k-j} + z^{-(k+j)}]\frac{dz}{z}$$

$$= \frac{1}{2\pi i} \oint \frac{[\sum_{s=-nc}^{nc} b_s z^{-s}]^2 z^{-(na+nc)}}{A^2(z^{-1})} [z^{k-j} + z^{-(k+j)}]\frac{dz}{z} .$$

APPENDIX B: PROOF OF LEMMA 5.2

Let

$$D \triangleq \left.\frac{\partial \hat{a}(\hat{\theta})}{\partial \hat{\theta}}\right|_{\hat{\theta}=\theta} . \qquad (B.1)$$

Then

$$P_m^a = D\, P_m^\theta\, D^T , \qquad (B.2)$$

where P_m^θ is given by (5.1). Some straightforward calculations give

$$\left.\frac{\partial \hat{a}}{\partial \hat{r}_k}\right|_{\hat{\theta}=\theta} = \left.\{\hat{R}^{-1} \frac{\partial \hat{R}}{\partial \hat{r}_k} \hat{R}^{-1}\hat{r} - \hat{R}^{-1} \frac{\partial \hat{r}}{\partial \hat{r}_k}\}\right|_{\hat{\theta}=\theta} = -R^{-1} \left.\frac{\partial}{\partial \hat{r}_k}\{\hat{R}\, a + \hat{r}\}\right|_{\hat{\theta}=\theta} . \qquad (B.3)$$

Thus, $D = -R^{-1}G$, where

$$G \triangleq \left.\frac{\partial}{\partial \hat{\theta}} \{\hat{R}\, a + \hat{r}\}\right|_{\hat{\theta}=\theta} . \qquad (B.4)$$

The (j,k) element of G is given by

$$G_{jk} = \left\{\frac{\partial}{\partial \hat{r}_k} \sum_{i=0}^{na} a_i \hat{r}_{nc+j-i}\right\}\Bigg|_{\hat{\theta}=\theta} =$$

$$= \begin{cases} \sum_{i=0}^{na} a_i \delta_{|nc+j-i|,k} = a_{nc+j+k} + a_{nc+j-k}, & k \neq 0, \\ \sum_{i=0}^{na} a_i \, \delta_{nc+j-i,0} = a_{nc+j}, & k = 0, \end{cases} \tag{B.5}$$

for $j=1,\ldots,na$, $k = 0,\ldots,na+nc$.

In (B.5) we have set $a_k = 0$ for $k > na$ and $k < 0$, and $a_0 = 1$.

Next we evaluate the matrix products GW_{12} and $GW_{11}G^T$ which appear in (B.2). The (i,j) element of GW_{12} is given by, cf. (4.4e),

$$[GW_{12}]_{ij} = E\{(\sum_{k=0}^{na+nc} G_{ik} A^2(q^{-1})[v(t-k) + v(t+k)])v(t-na-nc-j)\}, \tag{B.6}$$

$$, \quad i=1,\ldots,na, \quad j=1,\ldots,m-n\theta,$$

where cf. (B.5),

$$\sum_{k=0}^{na+nc} G_{ik}[v(t-k)+v(t+k)] = \sum_{k=-na-nc}^{na+nc} a_{nc+i+k} v(t-k) + \sum_{k=-na-nc}^{na+nc} a_{nc+i+k} \, v(t+k) =$$

$$= \sum_{s=0}^{na} a_s v(t-s+nc+i) + \sum_{s=0}^{na} a_s v(t+s-nc-i) =$$

$$= A(q^{-1})v(t+nc+i) + A(q) \, v(t-nc-i) .$$

It follows that

$$[GW_{12}]_{ij} = E\{\lambda A(q^{-1})C^2(q^{-1})e(t+nc+i)v(t-na-nc-j)\} +$$

$$+E\{A^2(q^{-1})v(t-nc-i)A(q^{-1})v(t-na-nc-j)\} =$$

$$= E\{A^2(q^{-1})v(t-i)A(q^{-1})v(t-na-j)\} . \tag{B.7}$$

The matrix $GW_{11}G^T$ could be evaluated by similar calculations. However, it is more convenient to note that $GW_{11}G^T$ is the covariance matrix of $\tilde{R}a + \tilde{r}$. In effect the following equality holds:

$$\tilde{R}a + \tilde{r} = G(\tilde{\theta} - \theta) . \tag{B.8}$$

The (i,j) element of $GW_{11}G^T$ is, therefore, given by, cf. (4.4c) and (B.8),

$$[GW_{11}G^T]_{ij} = \lim_{N\to\infty} N\, E\{ \sum_{k=0}^{na} a_k \tilde{r}_{nc+i-k}\} \cdot \{ \sum_{p=0}^{na} a_p \tilde{r}_{nc+j-p}\}$$

$$= \sum_{k=0}^{na} \sum_{p=0}^{na} a_k a_p [W_{11}]_{nc+i-k,nc+j-p} =$$

$$= \sum_{k=0}^{na} \sum_{p=0}^{na} a_k a_p E\{[v(t-nc-i+k) + v(t+nc+i-k)] \cdot [v(t-nc-j+p)]\} . \tag{B.9}$$

Denote the two terms in (B.9) by T_I and T_{II}.

$$T_I = \sum_{k=0}^{na} \sum_{p=0}^{na} a_k a_p E\{v(t-i-p)v(t-j-k)\} =$$

$$= E\{A(q^{-1})v(t-i)A(q^{-1})\, v(t-j)\} \tag{B.10}$$

$$T_{II} = \sum_{k=0}^{na} \sum_{p=0}^{na} a_k a_p E\{v(t-j)v(t+2nc+i-k-p)\} =$$

$$= E\{v(t-j)A^2(q^{-1})v(t+2nc+i)\} = E\{v(t-j) \cdot \lambda C^2(q^{-1})e(t+2nc+i)\} = 0 .$$

Thus, we have shown that

$$[GW_{11}G^T]_{ij} = E\{A(q^{-1})v(t-i)A(q^{-1})v(t-j)\} , \quad i,j=1,\dots,na . \tag{B.11}$$

The expression (5.9), (5.10) of P_m^a now readily follows from (5.1), (B.2), (B.3), (B.7) and (B.11). The inequality (5.11) is a simple consequence of (5.2) and (B.2). Finally, the expression of P_∞^a in (5.12) follows from (5.3), (B.2), (B.3), (B.11), the relation $\Gamma = G\Omega$ and some calculations similar to (B.6) - (B.7).

APPENDIX C: PROOF OF THEOREM 6.1

Let H denote the following nonsingular (m-nc-1) x (m-nc-1) matrix

$$H = \left[\begin{array}{ccccccc} 1 & & & & & & \\ 0 & \ddots & 1 & & 0 & & \\ \hline a_{na} & \cdots & a_1 & & 1 & & \\ & \ddots & & \ddots & & \ddots & \\ 0 & & \ddots & a_{na} & \cdots & a_1 & 1 \end{array}\right] \begin{array}{l} \left.\vphantom{\begin{array}{c}1\\0\end{array}}\right\} na \\ \left.\vphantom{\begin{array}{c}1\\0\\0\end{array}}\right\} m-n\theta \end{array} \tag{C.1}$$

Recall that $n\theta = na + nc + 1$. For $1 \le k \le m - n\theta$ and $1 \le j \le na$ we have

$$[a_{na}, \ldots, a_1, 1, 0, \ldots, 0] \cdot E\left\{\begin{bmatrix} y(t-nc-k) \\ \vdots \\ y(t-m) \end{bmatrix} y(t-j)\right\} =$$

$$\begin{aligned} &= E\{\sum_{i=0}^{na} a_{na-i}\, y(t-nc-k-i)y(t-j)\} = \\ &= E\{\sum_{p=0}^{na} a_p\, y(t-nc-k-na)y(t-j-p)\} = \\ &= E\{y(t-nc-na-k)C(q^{-1})e(t-j)\} = 0 \ . \end{aligned} \tag{C.2}$$

It follows that

$$H\, R_{m-nc-1} = \begin{bmatrix} R \\ 0 \end{bmatrix}, \tag{C.3}$$

with $R = R_{na}$ defined by (2.7).

Next we introduce the reciprocal polynomial of A(z)

$$A^*(z) \triangleq z^{na}A(z^{-1}) = a_{na} + a_{na-1}z + \ldots + a_1 z^{na-1} + z^{na} \ . \tag{C.4}$$

Then,

$$H\begin{bmatrix} y(t-1) \\ \vdots \\ y(t-m+nc+1) \end{bmatrix} = \begin{bmatrix} y(t-1) \\ \vdots \\ y(t-na) \\ \hline A^*(q^{-1})y(t-1) \\ \vdots \\ A^*(q^{-1})y(t-m+n\theta) \end{bmatrix},$$

which implies

$$HS_{m-nc-1}H^T = \lambda^2 E\{C(q^{-1}) \begin{bmatrix} y(t-1) \\ \vdots \\ y(t-na) \\ \hline A^*(q^{-1})y(t-1) \\ \vdots \\ A^*(q^{-1})y(t-m+n\theta) \end{bmatrix} \cdot$$

$$\cdot\ C(q^{-1})[y(t-1),\dots,y(t-na) \,\vdots\, A^*(q^{-1})y(t-1),\dots,\ A^*(q^{-1})y(t-m+n\theta)]\}. \quad (C.5)$$

Next note that

$$E\{C(q^{-1})A^*(q^{-1})y(t-i)C(q^{-1})y(t-j)\} =$$

$$= E\{\sum_{k=0}^{na} a_{na-k}C(q^{-1})y(t-i-k)C(q^{-1})y(t-j)\} = \quad (C.6)$$

$$= E\{\sum_{p=0}^{na} a_p\ C(q^{-1})y(t-i-na)C(q^{-1})y(t-j-p)\} =$$

$$= E\{C(q^{-1})y(t-i-na)A(q^{-1})C(q^{-1})y(t-j)\}\ ,$$

and similarly,

$$E\{C(q^{-1})A^*(q^{-1})y(t-i)C(q^{-1})A^*(q^{-1})y(t-j)\} =$$

$$= E\{ \sum_{k=0}^{na} \sum_{p=0}^{na} a_{na-k} a_{na-p} C(q^{-1}) y(t-i-k) C(q^{-1}) y(t-j-p)\} = \tag{C.7}$$

$$= E\{ \sum_{s=0}^{na} \sum_{\ell=0}^{na} a_s a_\ell \; C(q^{-1}) y(t-i-\ell) C(q^{-1}) y(t-j-s)\} =$$

$$= E\{A(q^{-1}) C(q^{-1}) y(t-i) A(q^{-1}) C(q^{-1}) y(t-j)\} \; .$$

By comparing with the definitions of Q_{11}, Q_{12} and W_{22} (see (5.10) and (4.4d)) we get

$$H \, S_{m-nc-1} H^T = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & W_{22} \end{bmatrix} . \tag{C.8}$$

It follows from (C.5) and (C.3) and (5.9) that

$$\tilde{P}_{m-nc-1} = \{R_{m-nc-1}^T H^T (H \, S_{m-nc-1} H^T)^{-1} H \, R_{m-nc-1}\}^{-1} =$$

$$= \left\{ [R^T, \; 0] \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & W_{22} \end{bmatrix}^{-1} \begin{bmatrix} R \\ 0 \end{bmatrix} \right\}^{-1} =$$

$$= R^{-1}(Q_{11} - Q_{12} \, W_{22}^{-1} \, Q_{12}^T) R^{-T} = P_m^a \quad ,$$

which concludes the proof.

APPENDIX D: A PARAMETRIZATION OF THE SPECTRAL DENSITY FUNCTION OF AN ARMA PROCESS

From the Yule-Walker equations, (2.4) it follows that ($a_0 \equiv 1$)

$$0 \equiv \sum_{k=1}^{\infty} \left(\sum_{j=0}^{na} a_j r_{nc+k-j} \right) z^{-k} = \sum_{j=0}^{na} a_j z^{-j} \sum_{k=1}^{\infty} r_{nc+k-j} z^{-(k-j)} =$$

$$= z^{nc} \sum_{j=0}^{na} a_j z^{-j} \sum_{p=nc+1-j}^{\infty} r_p z^{-p} = z^{nc} \sum_{j=0}^{na} a_j z^{-j} \left[\sum_{p=1}^{\infty} r_p z^{-p} - \sum_{p=1}^{nc-j} r_p z^{-p} \right] . \tag{D.1}$$

In establishing the last equality in (D.1) we tacitly assumed that

$$nc > na . \tag{D.2}$$

When (D.2) does not hold, the derivation needs to be modified and the following expressions become more complicated. Let

$$\tilde{\phi}(z) \triangleq \sum_{p=1}^{\infty} r_p z^{-p} . \tag{D.3}$$

Then

$$\phi(z) = r_0 + \tilde{\phi}(z) + \tilde{\phi}(z^{-1}) . \tag{D.4}$$

From (D.1) it follows that

$$A(z^{-1}) \tilde{\phi}(z) = \sum_{j=0}^{na} a_j \sum_{p=1}^{nc-j} r_p z^{-(p+j)} =$$

$$= \sum_{j=0}^{na} a_j \sum_{k=j+1}^{nc} r_{k-j} z^{-k} = \sum_{k=1}^{nc} r_k z^{-k} + \sum_{k=2}^{nc} a_1 r_{k-1} z^{-k} + \dots$$

$$\ldots + \sum_{k=na+1}^{nc} a_{na} r_{k-na} z^{-k} = (r_1)z^{-1} + (r_2 + a_1 r_1)z^{-2} +$$

$$\ldots + (r_{nc} + a_1 r_{nc-1} + \ldots + a_{na}\, r_{nc-na})z^{-nc}. \qquad (D.5)$$

Defining

$$p_k = \sum_{j=0}^{\min(k-1,na)} a_j r_{k-j} , \qquad k=1,\ldots,nc , \qquad (D.6)$$

we get

$$\tilde{\phi}(z) = \frac{\sum_{k=1}^{nc} p_k z^{-k}}{A(z^{-1})} . \qquad (D.7)$$

The spectral density $\phi(z)$ is, therefore, completely determined by

$$\theta' = [r_0, \ldots, r_{nc}; a_1, \ldots, a_{na}] , \qquad (D.8)$$

cf. (D.4)-(D.7). The parametrization via (D.8) of an ARMA process was used by Walker [11]. Cadzow [12] presented the explicit dependence of $\phi(z)$ on (D.8) as in (D.4), (D.6) and (D.7); see also [13]. Cadzow's derivation of (D.6), (D.7), however, provides less insight into the problem than the derivation above. Unfortunately, (D.6) and (D.7) rely on the assumption (D.2). If this assumption is not valid then $\tilde{\phi}(z)$ will have a more complicated expression than (D.6)-(D.7).

APPENDIX E: AN EXTENDED MAXIMUM LIKELIHOOD ESTIMATION PROBLEM AND ITS LARGE-SAMPLE SOLUTION

In this appendix we present a generalized version of the ML estimation problem introduced in section 3. The large-sample solution of the generalized ML problem can be obtained in a similar manner. The results of this appendix cover Walker's approach [11]. Even though these results are not used directly in the paper, we believe that they are useful in deriving new estimators in

some specific estimation problems.

Consider a random m-vector X which is asymptotically normally distributed such that for some $\bar{X}$ to be specified

$$\sqrt{N}(X-\bar{X}) \xrightarrow[N \to \infty]{\text{dist}} \mathcal{N}(0, W) , \tag{E.1}$$

where N denotes the length of the sample used to construct X. Let θ be the parameter vector to be estimated. Assume that θ completely determines the asymptotic distribution (E.1) of X. In contrast to the treatment in section 3 we now allow X to depend on θ. However, we impose some restrictions on this dependence. Thus, let X be partitioned as

$$X = \begin{bmatrix} x \\ z \end{bmatrix} \begin{matrix} \}n\theta \\ \end{matrix} , \qquad n\theta = \dim \theta .$$

We assume that

$$\bar{X} = \begin{bmatrix} \bar{x} \\ 0 \end{bmatrix} \begin{matrix} \}\ n\theta \\ \end{matrix} , \tag{E.2a}$$

where $x - \bar{x}$ has the form

$$x - \bar{x} = -B\theta + r , \tag{E.2b}$$

where B is a nonsingular (at least for $N \to \infty$) matrix, and where B and r depend on the data only. Furthermore, we assume that there exists $\hat{z}$ such that

$$|\hat{z} - z| = O(1/N) , \tag{E.3a}$$

and W such that

$$|\hat{W} - W| = O(1/\sqrt{N}) , \tag{E.3b}$$

where both $\hat{z}$ and $\hat{W}$ depend only on the data at hand.

Note that assumption (E.3b) is fairly weak. The matrix $\hat{W}$ may be taken as $W(\tilde{\theta})$ with $\tilde{\theta}$ a consistent estimate of θ . Similarly, (E.3a) will be satisfied by taking $\hat{z} = z(\tilde{\theta})$ provided that

$$\left|\frac{\partial}{\partial\theta} z(\theta)\right| = O(1/\sqrt{N}) \ . \tag{E.3c}$$

This can be seen from the following Taylor series expansion

$$z(\tilde{\theta}) = z(\theta) + [\frac{\partial}{\partial\theta} z(\theta)](\tilde{\theta}-\theta) + O(1/N) = z(\theta) + O(1/N) \ , \tag{E.4}$$

where the second equality follows from (E.3c). Satisfying the conditions (E.2) and (E.3c) in a given application requires careful choice of X (for a specific example see [11]).

Under the conditions above we derive a simple large-sample approximation of the ML estimate of θ , in the manner of section 3. The asymptotic log-likelihood function of X is given by (3.2). Paralleling the analysis in section 3 we obtain an asymptotically valid approximation of the derivative with respect to θ of the log-likelihood function:

$$\frac{1}{N}\frac{\partial L(\theta)}{\partial\theta} = [B, O(1/\sqrt{N})]W^{-1}(X-\bar{X}) + O(1/N) =$$

$$= [B,\ 0]\hat{W}^{-1}\begin{bmatrix}-B\theta+r\\ \hat{z}\end{bmatrix} + O(1/N) \ . \tag{E.5}$$

An approximation of order 1/N of the ML estimate (the solution of the equation $\partial L(\theta)/\partial\theta = 0$) is given by

$$[B,\ 0]\hat{W}^{-1}\begin{bmatrix}-B\theta+r\\ \hat{z}\end{bmatrix} = 0 \ . \tag{E.6}$$

Using (3.6), we rewrite (E.6) as

$$B\theta = r - \hat{W}_{12}\hat{W}_{22}^{-1}\hat{z}, \tag{E.7}$$

which gives

$$\hat{\theta} = B^{-1}(r - \hat{W}_{12}\hat{W}_{22}^{-1}\hat{z}) \ . \tag{E.8}$$

Concerning the asymptotic accuracy properties of $\hat{\theta}$, (E.8), we can prove results analogous to those of section 3. To save space we shall omit the details.

APPENDIX F: THE NONSINGULARITY OF W_{22} and $\hat{W}_{22}$.

In this appendix we analyze a condition which was tacitly assumed to hold. It was assumed that the inverse $(m-n\theta) \times (m-n\theta)$ matrices W_{22}^{-1} and $\hat{W}_{22}^{-1}$ exist. Since we let m tend to infinity this assumption should be analyzed with some care. Indeed, some eigenvalues of these matrices might tend to zero as $m \to \infty$ and then, even though the matrices are nonsingular for any finite m, they may be ill-conditioned for large m. To address these issues we state the following result.

<u>Lemma F.1.</u> Consider the mxm matrix W_{22} given by (4.4d). Let $\{\lambda_j, j=1,\ldots,m\}$ denote the eigenvalues of W_{22} and let

$$\lambda_{min}^{(m)} = \inf_j\{\lambda_j\} \ , \ \lambda_{max}^{(m)} = \sup_j\{\lambda_j\} \ . \tag{F.1}$$

Then

$$\lambda_{min}^{(m)} \geq \lambda_{min}^{(m+1)} \ , \ \lambda_{max}^{(m)} \leq \lambda_{max}^{(m+1)} \ , \tag{F.2}$$

and

$$\sigma_{min} \triangleq \lim_{m\to\infty} \lambda_{min}^{(m)} = \lambda^4 \inf_\omega |C(e^{i\omega})|^4, \tag{F.3a}$$

$$\sigma_{max} \triangleq \lim_{m\to\infty} \lambda_{max}^{(m)} = \lambda^4 \sup_\omega |C(e^{i\omega})|^4 \ . \tag{F.3b}$$

<u>Proof:</u> The inequalities in (F.2) are direct consequences of the fact that as m increases, the sequence of W_{22} matrices is a sequence of nested non-negative definite matrices. We will now prove (F.3a). (The proof of (F.3b) is

similar). Let γ be a real number, and consider the matrix $W_{22} - \gamma I$. The (k,p) element of this matrix is given by

$$\frac{\lambda^4}{2\pi} \int_{-\pi}^{\pi} |C(e^{i\omega})|^4 e^{i\omega(k-p)} d\omega - \gamma\delta_{k,p} = \qquad \text{(F.4)}$$
$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \{\lambda^4 |C(e^{i\omega})|^4 - \gamma\} e^{i\omega(k-p)} d\omega .$$

If γ is such that

$$\lambda^4 |C(e^{i\omega})|^4 - \gamma > 0 , \qquad \text{for } \omega \in (-\pi, \pi] , \qquad \text{(F.5)}$$

then it follows from (F.4) that $W_{22} - \gamma I$ is the covariance matrix of a moving-average process with a covariance generating function equal to the left-hand-side of (F.5). Thus

$$W_{22} > \gamma I , \qquad \text{for } \underline{\text{all}} \ m . \qquad \text{(F.6)}$$

If (F.5) does not hold, (F.6) cannot be true. Now, σ_{min} is uniquely defined by the following two conditions.

$$W_{22} > \sigma_{min} I , \qquad \text{for } \underline{\text{all}} \ m . \qquad \text{(F.7a)}$$

and

$$W_{22} > (\sigma_{min} + \varepsilon) I , \qquad \varepsilon > 0, \text{ cannot hold for all } m . \qquad \text{(F.7b)}$$

From the above discussion it readily follows that σ_{min} is given by (F.3b).

Since we assumed that C(z) has no zeros on the unit circle (A1) we conclude from the lemma above that $\sigma_{min} > 0$. Thus, W_{22}^{-1} exists for any value of m (finite or infinite). However, note that if the polynomial C(z) has zeros close to the unit circle then some numerical problems may be expected. Indeed in such a case σ_{min} will be small and then W_{22} will be ill-conditioned for large m, cf. (F.3). Since $\hat{W}_{22}$ is a consistent estimate of W_{22} we expect that the discussion above applies to $\hat{W}_{22}$ as well, provided that N is sufficiently large.

In the small-sample case some additional care may be needed. The matrix $\hat{W}_{22}$ is obtained from (4.4d) where $\{b_k\}$ are replaced by $\{\tilde{b}_k\}$ computed from $\{a_i\}$ and $\{\bar{r}_j\}$ via (2.5). When $C(z)$ has zeros close to the unit circle it may happen that the estimated symmetric polynomial

$$B(z) = \sum_{k=-nc}^{nc} \tilde{b}\, z^{-k} \tag{F.8}$$

has zeros <u>on</u> the unit circle. This is, for example, the case whenever $B(z)$ is not factorizable. As is known, the polynomial $B(z)$ will have in this case complex zeros with odd degree of multiplicity on the unit circle.For $\hat{W}_{22}$, (F.3a) becomes

$$\hat{\sigma}_{min} \triangleq \lim_{m\to\infty} \lambda^{m}_{min}(\hat{W}_{22})\sigma(\hat{W}_{22}) = \lambda^4 \inf_{\omega} |B(e^{j\omega})|^2 \tag{F.9}$$

Thus, if (F.8) has zeros on the unit circle then we get from (F.9) that $\hat{\sigma}_{min} = 0$ and, therefore, we expect $\hat{W}_{22}$ to be very ill-conditioned for large m. To avoid such cases we may need to determine the zeros of (F.8) and perform some correction on those which are on, or too close to, the unit circle.

APPENDIX H

MAXIMUM LIKELIHOOD ESTIMATION OF THE PARAMETERS OF MULTIPLE SINUSOIDS FROM NOISY MEASUREMENTS

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MAXIMUM LIKELIHOOD ESTIMATION OF THE PARAMETERS OF MULTIPLE SINUSOIDS FROM NOISY MEASUREMENTS

P. Stoica, R. Moses, B. Friedlander and T. Söderström

ABSTRACT

The problem of estimating the frequencies, phases, and amplitudes of sinusoidal signals is considered. A simplified maximum-likelihood Gauss-Newton algorithm which provides asymptotically efficient estimates of these parameters is proposed. Initial estimates for this algorithm are obtained by a variation of the overdetermined Yule-Walker method. Some numerical examples are presented to illustrate the performance of the proposed estimation procedure.

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1. INTRODUCTION

The problem of estimating the parameters of sinusoidal signals from noisy data has received considerable attention recently [1]. The sinusoid parameters can be estimated using correlation based techniques. These include Prony's method, Pisarenko's harmonic decomposition procedure, and the Yule-Walker method in one of its many versions.

Prony's Method (see [2] for a recent survey) is known to give inconsistent estimates. It cannot be used in cases with a low signal-to-noise ratio since the resulting estimates may be highly biased. In Pisarenko's procedure [2] this problem is eliminated. This method gives consistent estimates, but in some cases it has poor accuracy.

The basic Yule-Walker method [1],[2] does not eliminate this deficiency of Pisarenko's method. It gives consistent estimates, but its accuracy may be poor. Since the Yule-Walker method is attractive from the computational standpoint, much effort has been spent in recent years to improve its accuracy properties.

The overdetermined or high-order Yule-Walker method is a modification of the basic Yule-Walker procedure, which was reported to lead to a considerable increase in resolution [3],[4],[5],[6]. This method was proposed heuristically, and the properties of the corresponding estimates were analyzed by Monte-Carlo simulations only. The reasons for the increase of the parameter estimation accuracy when the number of Yule-Walker equations and the model order are increased, were not too well understood. In [11] and [12] we have tried to fill this gap. Very briefly, the conclusions of [11],[12] are that the asymptotic accuracy of the Yule-Walker estimates will increase with the number of Yule-Walker equations used and with the model order, although not necessarily monotonically. However, even when the number of Yule-Walker equations and the model order are increased without bound, the limiting accuracy may still be worse than that corresponding to the Cramér-Rao lower bound (CRLB). Thus, in general, it is possible to improve the accuracy of the Yule-Walker based estimates.

In this paper we consider the following procedure for estimating the parameters of sinusoids in noise. We use the overdetermined Yule-Walker (OYW) method to get initial estimates of the sinusoid parameters. These are then used as starting point in a Gauss-Newton algorithm for maximizing the likelihood function (under the assumption that the measurement noise is Gaussian). Since the OYW method provides good initial estimates, the Gauss-Newton algorithm needs relatively few iterations to converge. Also, the problem of convergence to local maxima is not likely to occur. Furthermore, we show a way to considerably simplify the Gauss-Newton algorithm. The simplified algorithm is also more stable from the numerical point of view. Yet it has the same convergence point and, at least asymptotically, the same convergence rate as the original Gauss-Newton algorithm. We show by means of a number of Monte-Carlo simulations that the (simplified) maximum-likelihood (ML) Gauss-Newton algorithm has better resolution than the OYW method. Comparisons with the asymptotic CRLB are also included.

Some studies related to the present paper were reported in [7] and [8]. In [7] an approximate ML method is discussed. A relatively simple numerical algorithm is obtained, at the cost of sacrificing some accuracy. The method proposed here is of comparable complexity, but has better asymptotic accuracy. Reference [8] presents a performance comparison of several estimation techniques based on linear prediction and on Singular Value Decomposition.

The outline of the paper is as follows. In section 2 we state the problem considered here. Section 3 contains a brief review of the overdetermined Yule-Walker method for estimating the sinusoid parameters. This method is used to provide initial estimates for the proposed maximum likelihood method, which is described in section 4. The asymptotic properties and some computational aspects of both methods (OYW and ML), are briefly discussed. The problem of local minima of the cost function being minimized in the proposed method, is discussed in section 5. Numerical examples illustrating the performance of the proposed technique are presented in section 6.

2. STATEMENT OF THE PROBLEM

Consider the following sinusoidal signal

$$x(t) = \sum_{i=1}^{m} \alpha_i \sin(\omega_i t + \phi_i) , \quad t=1,2,\dots , \tag{2.1a}$$

where

$$\alpha_i, \phi_i \in R , \ \omega_i \in (0, \pi) , \qquad \text{and} \quad \omega_i \neq \omega_j \qquad \text{for} \quad i \neq j . \tag{2.1b}$$

The assumption $\omega_i \neq 0$ means that a possible non-zero constant level of x(t) has been removed. The condition $\omega_i < \pi$ is a consequence of Shannon's sampling theorem.

Let y(t) denote the noise-corrupted measurements of x(t)

$$y(t) = x(t) + \varepsilon(t) , \tag{2.2}$$

where $\{\varepsilon(t)\}$ is a sequence of independent and identically distributed Gaussian random variables with zero mean and variance λ^2. We assume that x(t) and $\varepsilon(s)$ are independent for any t and s.

The assumption that $\varepsilon(t)$ is Gaussian may appear somewhat restrictive. Under the Gaussian hypothesis it is easy to write the likelihood function of the data and to obtain an explicit expression for the CRLB. If in some application the Gaussian hypothesis fails to be true, the algorithm of this paper is still applicable, but it will no longer provide the ML estimates. Nevertheless, the estimates obtained by using the algorithm will still give the minimum variance in a fairly large class of estimators whose covariance matrices depend only on the second order statistics of the data. This is explained further in section 4.

Next we denote by r_n the covariance of y(t) at lag n (n=0, 1,2,...)

$$r_n = E\{y(t)\ y(t-n)\} . \tag{2.3}$$

The operator $E\{\cdot\}$ denotes statistical expectation. The sample covariances corresponding to (2.3) shall be denoted by $\hat{r}_n$. We will use the following definition of $\hat{r}_n$

$$\hat{r}_n = \frac{1}{N-n} \sum_{t=1}^{N-n} y(t)\, y(t+n) \ , \quad n = 0,1,2,\ldots$$
$$\hat{r}_{-n} = \hat{r}_n \ , \tag{2.4}$$

where N denotes the length of the data sample.

Collecting the amplitudes $\{\alpha_i\}$, phases $\{\phi_i\}$ and frequencies $\{\omega_i\}$ in a single parameter vector, we define

$$\theta = [\alpha_1,\ldots,\alpha_m,\ \phi_1,\ldots,\phi_m,\ \omega_1,\ldots,\omega_m]^T \ . \tag{2.5}$$

The problem considered in this paper is the estimation of θ from N samples of noisy measurements $\{y(1),\ldots,y(N)\}$.

3. THE INITIAL OVERDETERMINED YULE-WALKER ESTIMATES

As is well known $x(t)$, (2.1), obeys a homogeneous difference equation of order 2m,

$$x(t) + a_1 x(t-1) + \dots + a_n x(t-n) = 0 \ , \quad n \triangleq 2m \ , \tag{3.1}$$

where $\{a_i\} \varepsilon R$ are such that the polynomial

$$A(z) = 1 + a_1 z + \dots + a_n z^n \ , \tag{3.2a}$$

has all its zeros located on the unit circle at $e^{\pm i\omega_k}$, i.e.,

$$A(e^{\pm i\omega_k}) = 0 \quad , \qquad k = 1, \dots, m \ . \tag{3.2b}$$

See [2],[4],[5],[13]. Since we have

$$r_n = E\{x(t)\ x(t+n)\} + \lambda^2\ \delta_{n,0} \ , \tag{3.3a}$$

where $\delta_{i,j}$ is the Dirac delta

$$\delta_{i,j} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases} \ , \tag{3.3b}$$

it follows from (3.1) that the coefficients $\{a_i\}$ obey the so-called (modified) Yule-Walker equations

$$\sum_{i=0}^{n} a_i r_{n+k-i} = 0 \ , \qquad k \geq 1 \quad , \quad (a_0 = 1) \quad , \tag{3.4}$$

A commonly used technique for estimating the frequencies $\{\omega_i\}$ is based on (3.4). Consistent estimates $\{\hat{a}_i\}$ can be obtained by solving the following linear system of equations.

$$\begin{bmatrix} \hat{r}_n & \cdots & \hat{r}_1 \\ \hat{r}_{n+1} & \cdots & \hat{r}_2 \\ \vdots & & \vdots \\ \hat{r}_{L-1} & \cdots & \hat{r}_{L-n} \end{bmatrix} \begin{bmatrix} \hat{a}_1 \\ \vdots \\ \hat{a}_n \end{bmatrix} = - \begin{bmatrix} \hat{r}_{n+1} \\ \hat{r}_{n+2} \\ \vdots \\ \hat{r}_L \end{bmatrix}, \quad L > 2n, \tag{3.5}$$

where $\{\hat{r}_i\}$ are the sample covariances. The matrix appearing in (3.5) has full rank, at least for large N, [14]. Note that for $L > 2n$ the system (3.5) is overdetermined and needs to be solved in a least-squares sense. Intuitively we can expect that the larger L, the more accurate will be the estimates $\{\hat{a}_i\}$, since the covariances for large lags contain "useful information" about the covariance structure of the data. While it is not always true that increasing L increases estimation accuracy [12], it was shown by simulations [3],[6], that increasing L is often useful. A theoretical explanation of this empirically noticed fact was recently presented in [12]. It was shown there that while the asymptotic (for $N \to \infty$) accuracy of $\{\hat{a}_i\}$, does not increase monotonically with L, it improves considerably in the limit as $L \to \infty$. For $L < \infty$ the estimation errors $(\hat{a}_i - a_i)$ are of order $1/\sqrt{N}$, and for $L \to \infty$ they are of order $1/L\sqrt{N}$. The estimation technique based on (3.5) with $L > 2n$ is the so-called overdetermined Yule-Walker (OYW) method [3]-[6].

The frequencies $\{\omega_i\}$ can now be estimated by determining the roots of

$$\hat{A}(z) = 1 + \hat{a}_1 z + \ldots + \hat{a}_n z^n = 0 \tag{3.6}$$

Note that determining the estimates $\{\hat{\omega}_i\}$ from (3.6) implies, in general, some approximations since $\hat{A}(z)$ is not guaranteed to have all of its zeros on the unit circle. (For example, one may look at the peaks of $1/|\hat{A}(e^{j\omega})|^2$, or at the angles of the roots of $\hat{A}(z)$).

The problem of determining estimates of $\{\alpha_i\}$ and $\{\phi_i\}$ once estimates $\{\hat{\omega}_i\}$ of the frequencies are given, can be reduced to a least-squares fit. Rewrite (2.1), (2.2) as

$$y(t) = \sum_{k=1}^{m} (\beta_k \sin \omega_k t + b_k \cos \omega_k t) + \varepsilon(t) , \tag{3.7a}$$

where

$$\beta_k = \alpha_k \cos \phi_k , \qquad b_k = \alpha_k \sin \phi_k . \tag{3.7b}$$

Replacing $\{\omega_i\}$ in (3.7) by their estimates $\{\hat{\omega}_i\}$, the problem of estimating $\beta_k, b_k\}$ can be formulated as the following minimization problem:

$$\min_{\{\beta_k, b_k\}} \sum_{t=1}^{M} \{y(t) - \sum_{k=1}^{m} (\beta_k \sin \hat{\omega}_k t + b_k \cos \hat{\omega}_k t)\}^2 , \qquad M < N . \tag{3.8}$$

The solution to this problem is given by

$$\hat{\psi} \triangleq \begin{bmatrix} \hat{\beta}_1 \\ \vdots \\ \hat{\beta}_m \\ \hat{b}_1 \\ \vdots \\ \hat{b}_m \end{bmatrix} = \{\frac{1}{M} \sum_{t=1}^{M} V(t)V(t)^T\}^{-1} \{\frac{1}{M} \sum_{t=1}^{M} V(t)y(t)\} , \tag{3.9a}$$

where

$$V(t) \triangleq [\sin\hat{\omega}_1 t, \ldots, \sin\hat{\omega}_m t, \cos\hat{\omega}_1 t, \ldots, \cos\hat{\omega}_m t]^T . \tag{3.9b}$$

The reason for not using all of the N data points in (3.8), (3.9) will be explained later. It will be shown that if M in (3.8) is too large (e.g. M=N) then the estimation accuracy may deteriorate considerably. Note that for $M < N$ we also get a smaller computational burden.

Using $\{\hat{\beta}_i\}$ and $\{\hat{b}_i\}$ in (3.7b) we readily obtain estimates of $\{\alpha_i\}$ and $\{\phi_i\}$ as given by

$$\begin{aligned} \hat{\phi}_j &= \mathrm{arctg}\{\hat{b}_j/\hat{\beta}_j\} \quad (\mathrm{mod}\ 2\pi), \\ & \qquad\qquad\qquad\qquad j = 1, \ldots, m . \\ \hat{\alpha}_j &= \hat{\beta}_j/\cos \hat{\phi}_j \end{aligned} \tag{3.10}$$

Next we discuss some implementation issues related to (3.9). Straightforward programming of (3.9) would lead to a large computational burden. The main reason is that calculation of trigonometric functions on a computer is time-consuming. Note, however, that the solution $c_i(t)$ of the following second-order difference equation

$$c_i(t) - (2\cos\omega_i)\, c_i(t-1) + c_i(t-2) = 0 , \qquad t=3,4,...$$

with initial conditons

$$c_i(1) = \cos \omega_i , \quad c_i(2) = \cos 2\, \omega_i , \tag{3.11b}$$

is given by

$$c_i(t) = \cos \omega_i t , \qquad t=1,2,... \tag{3.11c}$$

A different set of initial conditions $(c_i(1) = \sin\omega_i,\ c_i(2) = \sin 2\omega_i)$ will produce $c_i(t) = \sin \omega_i t$. Thus, the sequences $\{\sin\omega_i t,\ \cos\omega_i t;\ t=1,...,M;\ i=1,..,m\}$ can be generated using (3.11) at a cost of approximately 2mM multiplications, and the vector $\Sigma V(t)y(t)$ in (3.9) will require a total of 4mM multiplications.

Next we present an efficient way for computing the matrix $\Sigma V(t)V(t)^T$ in (3.9). It follows from Lemma A.1 in the Appendix that

$$\frac{1}{M}\sum_{t=1}^{M} \sin \omega_i t \sin \omega_j t = \frac{1}{2M}\sum_{t=1}^{M}[\cos \omega_{ij}^- t - \cos \omega_{ij}^+ t]$$

$$= \frac{1}{2M}\left\{\frac{\sin[\frac{M\omega_{ij}^-}{2}]\cos[\frac{(M+1)\omega_{ij}^-}{2}]}{\sin[\frac{\omega_{ij}^-}{2}]} - \frac{\sin[\frac{M\omega_{ij}^+}{2}]\cos[\frac{(M+1)\omega_{ij}^+}{2}]}{\sin[\frac{\omega_{ij}^+}{2}]}\right\} , \tag{3.12}$$

where

$$\omega_{ij}^- = \omega_i - \omega_j , \qquad \omega_{ij}^+ = \omega_i + \omega_j .$$

Similar expressions can be derived for the other elements of the matrix in (3.9). For large M we can further simplify the computations by using some approximations. From Lemma A.1 it follows that

$$\frac{1}{M}\sum_{t=1}^{M} V(t)V(t)^T = \frac{1}{2} I_{2m} + O(\frac{1}{M}) \tag{3.13}$$

In (3.13) we tacitly assumed that $\hat{\omega}_i \neq \hat{\omega}_j$ for $i \neq j$. If this is not the case, we can work with $\hat{\omega}_i$ and $\hat{\omega}_j$ slightly corrected as

$$\hat{\omega}_i \leftarrow \hat{\omega}_i - \varepsilon\ , \quad \hat{\omega}_j \leftarrow \hat{\omega}_j + \varepsilon\ , \tag{3.14}$$

for some ε of order $1/L\sqrt{N}$. We conclude from (3.13) that for large M the following simple estimate

$$\tilde{\psi} = \frac{2}{M}\sum_{t=1}^{M} V(t)y(t), \tag{3.15}$$

is an approximation of order 1/M of $\hat{\psi}$, (3.9). Note, however, that the smaller $\inf_{i \neq j}|\hat{\omega}_i - \hat{\omega}_j|$ the larger the value of M needed for the approximation in (3.13) to be valid (see the discussion in the appendix and also equation (3.12)). If M is not large enough then $\tilde{\psi}$ may not be a good approximation of $\hat{\psi}$. Furthermore, the calculation of $\hat{\psi}$ may be problematic in such a case since the matrix in (3.9) will be ill-conditioned.

We conclude this section with a discussion of the asymptotic properties of the estimates introduced above. The frequency estimates $\{\hat{\omega}_i\}$ obtained by the OYW method are consistent, [15]. The asymptotic (as $N, L \rightarrow \infty$) standard deviations of $\{\omega_i - \omega_i\}$ are of order $1/L\sqrt{N}$, provided that L increases not faster than N^γ, with $\gamma < 1/2$ [12]. The condition $\gamma < 1/2$ is sufficient but probably not necessary. A necessary and sufficient condition on γ is not known. Since the CRLB on the standard deviation of $\{\hat{\omega}_i\}$ is $O(1/N^{3/2})$ as is shown in the appendix, it seems possible to improve significantly the accuracy of the OYW estimates.

An analysis of the asymptotic behavior of $\{\hat{\alpha}_i, \hat{\phi}_j\}$, (3.9), (3.10), does not seem to be available in literature. Due to the use in (3.9) of $\{\hat{\omega}_i\}$ instead of $\{\omega_i\}$ such an analysis is not so easy. Since $\{\hat{\alpha}_i\}$ and $\{\hat{\phi}_j\}$ are used as initial estimates, their accuracy is not too important, and will not be discussed in detail. What is, however, quite important is the choice of M in (3.9). To simplify notation, we will consider the case of a single sinusoid (m=1). It should be emphasized, however, that the same conclusions apply also for m > 1.

For m=1 and large M we have from (3.9), (3.13),

$$\hat{\psi}-\psi = \frac{2}{M}\sum_{t=1}^{M}\begin{bmatrix}\sin\hat{\omega}t\\ \cos\hat{\omega}t\end{bmatrix}\{y(t)-[\sin\hat{\omega}t\ \cos\hat{\omega}t]\psi\} + O(\tfrac{1}{M}) =$$

$$= \frac{2}{M}\sum_{t=1}^{M}\begin{bmatrix}\sin\omega t\\ \cos\omega t\end{bmatrix}\varepsilon(t) + \frac{2}{M}\sum_{t=1}^{M}\left\{\begin{bmatrix}t\cos\omega t\\ -t\sin\omega t\end{bmatrix}\varepsilon(t) - \right.$$

$$\left.\begin{bmatrix}\sin\omega t\\ \cos\omega t\end{bmatrix}[t\cos\omega t\ \ -t\sin\omega t]\psi\right\}(\hat{\omega}-\omega) +$$

$$+ \frac{1}{M}\sum_{t=1}^{M}\left\{\begin{bmatrix}-t^2\sin\omega t\\ -t^2\cos\omega t\end{bmatrix}\varepsilon(t) - 2\begin{bmatrix}t\cos\omega t\\ -t\sin\omega t\end{bmatrix}[t\cos\omega t\ -t\sin\omega t]\psi +\right.$$

$$\left.+\begin{bmatrix}\sin\omega t\\ \cos\omega t\end{bmatrix}[t^2\sin\omega t,\ t^2\cos\omega t]\psi\right\}(\hat{\omega}-\omega)^2 + \ldots + O(1/M)\ , \qquad (3.16)$$

where $\psi = [\beta,\ b]^T$ is the vector of the true parameters. It is not difficult to see that the first term in (3.16) is $O(1/\sqrt{M})$. Since $\hat{\omega}-\omega = O(1/L\sqrt{N})$, see the discussion above, it can be shown that the second term is $O(M/L\sqrt{N})$, the third is $O(M^2/(L\sqrt{N})^2)$, etc. Thus if M increases faster than $L\sqrt{N}$ (for example, if we set $L = N^{1/2-\delta}$ for some $\delta > 0$, and M=N), then difficulties

may occur. Indeed, in such a case the estimate $\hat{\psi}$ may not be consistent. The condition $M \ll L\sqrt{N}$ must be imposed. Then the first and second terms in (3.16) are asymptotically the dominant ones. Note that the magnitude of the first term decreases with M while that of the second increases with M. To get good asymptotic properties for $\hat{\psi}$ (i.e., small estimation error $\hat{\psi}-\psi$), M should be chosen such that these two terms have the same magnitude. Thus the "optimal" rate of increase of M is given by

$$M = (L\sqrt{N})^{2/3} . \tag{3.17}$$

The estimation error $(\hat{\psi}-\psi)$ corresponding to this choice of M, is of the order $1/\sqrt{M}$.

4. A MAXIMUM LIKELIHOOD ALGORITHM

The ML estimate of θ is obtained as the minimum point of the following loss function (see [7],[8] and also the appendix)

$$LF = \sum_{t=1}^{N} \varepsilon^2(t,\theta) , \tag{4.1a}$$

where

$$\varepsilon(t,\theta) = y(t) - \sum_{i=1}^{m} \alpha_i \sin(\omega_i t + \phi_i) , \tag{4.1b}$$

We use the Gauss-Newton algorithm to minimize (4.1). Let $\hat{\theta}^k$ denote the parameter estimate at iteration k. The updated estimate $\hat{\theta}^{k+1}$ is computed by

$$\hat{\theta}^{k+1} = \hat{\theta}^k - [\sum_{t=1}^{N} \varepsilon_\theta(t,\hat{\theta}^k)\varepsilon_\theta^T(t,\hat{\theta}^k)]^{-1}[\sum_{t=1}^{N} \varepsilon_\theta(t,\hat{\theta}^k)\varepsilon(t,\hat{\theta}^k)] \tag{4.2a}$$

where

$$\varepsilon_\theta(t,\theta) \triangleq \frac{\partial\varepsilon(t,\theta)}{\partial\theta} , \tag{4.2b}$$

and where we set

$$\hat{\theta}^0 = \hat{\theta} \triangleq \text{the OYW estimate.} \tag{4.2c}$$

The elements of the gradient vector $\varepsilon_\theta(t,\theta)$ are given by

$$\left.\begin{aligned} \frac{\partial\varepsilon(t,\theta)}{\partial\alpha_i} &= -\sin(\omega_i t + \phi_i) \\ \frac{\partial\varepsilon(t,\theta)}{\partial\phi_i} &= -\alpha_i \cos(\omega_i t + \phi_i) \\ \frac{\partial\varepsilon(t,\theta)}{\partial\omega_i} &= -t\, \alpha_i \cos(\omega_i t + \phi_i) \end{aligned}\right\} i=1,\ldots,m . \tag{4.3}$$

The matrix to be inverted in (4.2) contains entries of very different magnitudes. The elements of its left-top mxm block are of order N, while those of the right-bottom mxm block are of the order N^3 . Thus, it is desirable from the numerical standpoint to "balance" the elements of the

matrix. This will also be convenient for some subsequent theoretical considerations.

Let us introduce the notation

$$K_N = \begin{bmatrix} N^{1/2} I_{2m} & 0 \\ 0 & N^{3/2} I_m \end{bmatrix} \tag{4.4}$$

where I_K denotes the K x K identity matrix. The following recursion is equivalent to, but numerically more reliable than, (4.2a)

$$K_N \hat{\theta}^{k+1} = K_N \hat{\theta}^k - H_N^{-1}(\hat{\theta}^k)[K_N^{-1} \sum_{t=1}^{N} \varepsilon_\theta(t,\hat{\theta}^k)\varepsilon(t,\hat{\theta}^k)] \ , \tag{4.5a}$$

where

$$H_N(\theta) = K_N^{-1}[\sum_{t=1}^{N} \varepsilon_\theta(t,\theta)\varepsilon_\theta^T(t,\theta)]K_N^{-1} \ . \tag{4.5b}$$

Evaluation of the vector $\sum_{t=1}^{N} \varepsilon_\theta(t,\hat{\theta}^k)\varepsilon(t,\hat{\theta}^k)$ is straightforward. Its elements contain trigonometric functions which could be computed efficiently by the technique discussed in the previous section. Evaluation of the matrix $H_N(\hat{\theta}^k)$ can be done similarly but it appears quite costly. To overcome this difficulty we propose an approximate version of the iterative algorithm (4.2a).

As is shown in the Appendix,

$$H_N^{-1}(\theta) = G(\theta) + O(1/N) \ , \tag{4.6}$$

where

$$
G(\theta) = 2\left[\begin{array}{c|ccc|ccc}
I_m & & 0 & & & 0 & \\
\hline
 & \frac{4}{\alpha_1^2} & & 0 & -\frac{6}{\alpha_1^2} & & 0 \\
0 & & \ddots & & & \ddots & \\
 & 0 & & \frac{4}{\alpha_m^2} & 0 & & -\frac{6}{\alpha_m^2} \\
\hline
 & -\frac{6}{\alpha_1^2} & & 0 & \frac{12}{\alpha_1^2} & & 0 \\
0 & & \ddots & & & \ddots & \\
 & 0 & & -\frac{6}{\alpha_m^2} & 0 & & \frac{12}{\alpha_m^2}
\end{array}\right]. \tag{4.7}
$$

Replacing $H_N^{-1}(\hat{\theta}^k)$ in (4.5) by its large sample approximation $G(\hat{\theta}^k)$ we get,

$$
K_N\hat{\theta}^{k+1} = K_N\hat{\theta}^k - \mu_k\, G(\hat{\theta}^k)[K_N^{-1}\sum_{t=1}^{N} \varepsilon_\theta(t,\hat{\theta}^k)\varepsilon(t,\hat{\theta}^k)] \tag{4.8}
$$

where $\{\mu_k\}$ is a sequence of positive scalars which can be used for controlling the step size (μ_k can be determined, for example, by using a line search algorithm.). The algorithm (4.8) is much simpler than (4.2a). The two algorithms have clearly the same convergence point. Furthermore, for large N they will also have similar convergence rates.

We conclude this section by a discussion of the asymptotic accuracy of the limiting (as $k \to \infty$) estimate obtained by (4.8). Let this estimate be denoted by $\bar{\theta}$,

$$
\bar{\theta} = \lim_{k\to\infty} \hat{\theta}^k. \tag{4.9}
$$

Since we initialize the recursion (4.8) with a consistent estimate, it is expected to converge in a few iterations. In fact, parallelling the calculations in the proof of Theorem 4.1 below it is possible to show that (4.8) will asymptotically (as $N \to \infty$) converge in one iteration provided that L in (3.5) tends to infinity faster than $\sqrt{N}$.

Under the Gaussian hypothesis, $\bar{\theta}$ is the ML estimate. We expect, therefore, that its asymptotic covariance matrix equals the CRLB $P_{CR}^{\theta} = \lambda^2 G(\theta)$, see the appendix for the derivation of P_{CR}^{θ} . However, this

does not follow immediately since some of the standard assumptions of ML theory [10] fail to hold in our case (e.g. $\varepsilon_\theta(t,\theta)$ is a nonstationary process).

If we relax the Gaussian hypothesis, then $\tilde{\theta}$ is the prediction error (PE) estimate, [16]. Again, the standard PE theory does not apply to our problem. If it were applicable it would follow from [16] that the asymptotic covariance matrix of $\tilde{\theta}$ is still given by P_{CR}^{θ} .

The asymptotic covariance matrix of the normalized estimation errors $K_N(\tilde{\theta}-\theta)$ is derived next. We show that this matrix equals P_{CR}^{θ} , for any distribution function of the data.

<u>Theorem 4.1.</u> Consider the process $y(t)$ generated by (2.1),(2.2) under the assumptions stated except that $\varepsilon(t)$ is allowed to be non-Gaussian. Let $\tilde{\theta}$ be the estimate given by (4.9). Then

$$\lim_{N\to\infty} E[K_N(\tilde{\theta}-\theta)(\tilde{\theta}-\theta)^T K_N] = P_{CR}^{\theta}, \qquad (4.10)$$

where K_N is defined in (4.4) and $P_{CR}^{\theta} = \lambda^2 G(\theta)$.

<u>Proof</u>:

Note that,

$$K_N^{-1} \sum_{t=1}^{N} \varepsilon_\theta(t,\tilde{\theta})\varepsilon(t,\tilde{\theta}) = 0 \ . \qquad (4.11)$$

Thus, for large N we can write

$$\begin{aligned} 0 = {} & K_N^{-1} \sum_{t=1}^{N} \varepsilon_\theta(t,\theta)\varepsilon(t) + F(\theta)\, K_N(\tilde{\theta}-\theta) + \\ & + \frac{1}{2} \sum_{i=1}^{3m} (\tilde{\theta}_i-\theta_i)\, \frac{\partial F(\theta)}{\partial \theta_i} K_N(\tilde{\theta}-\theta) + \dots, \end{aligned} \qquad (4.12)$$

where θ_i is the i-th component of θ and

$$F(\theta)= K_N^{-1} \sum_{t=1}^{N} \{\varepsilon_\theta(t,\theta)\, \varepsilon_\theta^T(t,\theta)+\varepsilon_{\theta\theta}(t,\theta)\varepsilon(t,\theta)\}K_N^{-1}. \tag{4.13}$$

The first term in (4.12) is asymptotically independent of N. To see this note that its asymptotic covariance matrix, say P, is given by (see (4.5b) and (4.6))

$$P \triangleq \lim_{N\to\infty} E\{K_N^{-1}[\sum_{t=1}^{N} \sum_{s=1}^{N} \varepsilon_\theta(t,\theta)\varepsilon_\theta^T(s,\theta)\varepsilon(t)\varepsilon(s)]K_N^{-1}\} =$$

$$= \lambda^2 \lim_{N\to\infty} K_N^{-1}[\sum_{t=1}^{N} \varepsilon_\theta(t,\theta)\varepsilon_\theta^T(t,\theta)]K_N^{-1} = \lambda^4(P_{CR}^\theta)^{-1}, \tag{4.14}$$

where P_{CR}^θ is defined in the Appendix. The last equality in (4.14) is also proven in the Appendix.

Next we show that for large N

$$K_N^{-1}[\sum_{t=1}^{N} \varepsilon_{\theta\theta}(t,\theta)\varepsilon(t)]K_N^{-1}= O(1/\sqrt{N}). \tag{4.15}$$

The matrix $\varepsilon_{\theta\theta}(\cdot,\cdot)$ of second-order derivatives is given by

$$\varepsilon_{\theta\theta}(t,\theta)=\left[\begin{array}{c|c|c} 0 & -\mathrm{diag}[\cos(\omega_i t+\phi_i)] & -\mathrm{diag}[t\cos(\omega_i t+\phi_i)] \\ \hline -\mathrm{diag}[\cos(\omega_i t+\phi_i)] & \mathrm{diag}[\alpha_i \sin(\omega_i t+\phi_i)] & \mathrm{diag}[t\alpha_i \sin(\omega_i t+\phi_i)] \\ \hline -\mathrm{diag}[t\cos(\omega_i t+\phi_i)] & \mathrm{diag}[t\alpha_i \sin(\omega_i t+\phi_i)] & \mathrm{diag}[t^2\alpha_i \sin(\omega_i t+\phi_i)] \end{array}\right], \tag{4.16}$$

where each block of the matrix has size mxm. The generic element of the matrix in (4.15) can therefore be written as

$$V_N \triangleq \frac{\alpha}{N^{\beta+1}} \sum_{t=1}^{N} t^\beta \sin(\omega t+\phi)\varepsilon(t) , \tag{4.17}$$

where $\beta = \{0,1 \text{ or } 2\}$, $\alpha=\{\pm\alpha_i \text{ or } \pm 1\}$, $\omega = \omega_i$, and $\phi = \{\phi_i \text{ or } \phi_i+ \frac{\pi}{2}\}$. The variance of V_N is readily evaluated:

$$E\{V_N^2\} = \frac{\alpha^2}{N^{2\beta+2}} E\{\sum_{t=1}^{N} \sum_{s=1}^{N} t^\beta s^\beta \sin(\omega t + \phi)\sin(\omega s + \phi)\varepsilon(t)\varepsilon(s)\} =$$

$$= \frac{\alpha^2\lambda^2}{N^{2\beta+2}} \sum_{t=1}^{N} t^{2\beta}\sin^2(\omega t + \phi) \ .$$

Thus,

$$E\{V_N^2\} < \text{const.} \frac{1}{N^{2\beta+2}} \sum_{t=1}^{N} t^{2\beta} = O(1/N) \ ,$$

which proves (4.15).

It follows from the calculations above and from the Appendix that

$$F(\theta) = \lambda^2 (P_{CR}^{\theta})^{-1} + O(1/\sqrt{N}) \ .$$

Next we show that the higher-order terms in (4.12) can be neglected asymptotically. Note that

$\frac{\partial F(\theta)}{\partial \theta_i}$ is of the same order of magnitude as $F(\theta)$, for $i=1,\dots,2m$,

$\frac{\partial F(\theta)}{\partial \theta_i}$ is of the order of magnitude of $F(\theta)\cdot N$, for $i=2m+1,\dots,3m$.

Since

$$(\bar{\theta}_i - \theta_i) = \begin{cases} O(1/\sqrt{N}) & , \quad i=1,\dots,2m \ , \\ O(1/N\sqrt{N}) & , \quad i=2m+1,\dots,3m \ , \end{cases}$$

and since $F(\theta)$ is asymptotically independent of N as shown above, we conclude that the higher-order terms in (4.12) are $O(1/\sqrt{N})$. Thus, for large N,

$$K_N(\bar{\theta}-\theta) = -\frac{1}{\lambda^2} P_{CR}^{\theta} \cdot K_N^{-1} \sum_{t=1}^{N} \varepsilon_\theta(t,\theta)\varepsilon(t)$$

which implies that, cf (4.14)

$$\lim_{N\to\infty} E\{K_N(\tilde{\theta}-\theta)(\tilde{\theta}-\theta)^T K_N\} = \frac{1}{\lambda^2} P_{CR}^{\theta} \cdot \lambda^4 (P_{CR}^{\theta})^{-1} \cdot \frac{1}{\lambda^2} P_{CR}^{\theta} = P_{CR}^{\theta} .$$

It follows from the result above that in the case of Gaussian data, the estimate $\bar{\theta}$ of θ is asymptotically efficient. For non-Gaussian data, $\bar{\theta}$ will asymptotically be the minimum-variance estimate in a fairly large class of estimators whose covariance matrices depend only on the second-order statistics of the data.

5. THE PROBLEM OF LOCAL MINIMA

A major concern in any iterative minimization algorithm is the presence of local minima in the function to be minimized. Below we analyze the shape of the Loss Function (LF). For an arbitrary parameter vector $\hat{\theta}$ we can express $LF(\hat{\theta})$ as,

$$LF(\hat{\theta}) = N \cdot \{LF_s(\hat{\theta}) + LF_n(\hat{\theta}) + R\} \tag{5.1a}$$

where

$$LF_s(\hat{\theta}) = \frac{1}{N}\sum_{t=1}^{N}[x(t) - \hat{x}(t)]^2 \tag{5.1b}$$

$$LF_n(\underline{\hat{\theta}}) = \frac{\lambda}{N}\sum_{t=1}^{n} \varepsilon(t)[x(t) - \hat{x}(t)] \tag{5.1c}$$

$$R = \frac{1}{N}\sum_{t=1}^{n} \varepsilon^2(t) \tag{5.1d}$$

and where $\hat{x}(t)$ is defined as in (2.1) but with elements of $\hat{\theta}$ replacing elements of θ there. Comparing (5.1c) and (4.17), we see that $LF_n(\hat{\theta})$ is $O(1/\sqrt{N})$. Also, writing out (5.1a) and using (3.12) (with M=N), it is easy to show that

$$\begin{aligned} LF_s(\hat{\theta}) &= \sum_{i=1}^{m} \left(\frac{1}{N}\sum_{t=1}^{N}[\alpha_i\sin(\omega_i t + \phi_i) - \hat{\alpha}_i\sin(\hat{\omega}_i t + \hat{\phi}_i)]^2\right) + O(1/N) \\ &= \sum_{i=1}^{n} F_i(\hat{\alpha}_i, \hat{\omega}_i, \hat{\phi}_i) + O(1/N) \end{aligned}$$

where

$$F_i(\hat{\alpha}_i, \hat{\omega}_i, \hat{\phi}_i) = \frac{1}{N}\sum_{t=1}^{N}[\alpha_i\sin(\omega_i t + \phi_i) - \hat{\alpha}_i\sin(\hat{\omega}_i t + \hat{\phi}_i)]^2$$

Thus, to within $O(1/N)$ $LF_s(\hat{\theta})$ is the sum of m decoupled functions F_i; moreover, all of the F_i's have the same form. Understanding the shape of LF_s asymptotically reduces to understanding the shape of the function

$$F(\hat{\alpha}, \hat{\omega}, \hat{\phi}) = \frac{1}{N}\sum_{t=1}^{N}[\alpha\sin(\omega t + \phi) - \hat{\alpha}\sin(\hat{\omega}t + \hat{\phi})]^2 \tag{5.3}$$

It is easy to check that F is quadratic in $\hat{\alpha}$ and sinusoidal in $\hat{\phi}$, thus, the local minima of F with respect to these two variables are the global minima. However, F is not so well-behaved as a function of $\hat{\omega}$. A plot of $F(\hat{\omega})$ for $N=40$, $\omega=0.4$, $\phi = \hat{\phi} = 0$, and $\alpha = \hat{\alpha} = 1$ is shown in Figure 5.1. From this figure it is apparent that the initial estimate of $\hat{\omega}$ must be within the deep valley if we expect the Gauss-Newton algorithm to find the global minimum.

In Appendix B we show that the width of the valley is in the range $\Delta\omega\varepsilon[2\pi/N, 8\pi/N]$. Thus, the initial estimate must have a standard deviation on the order of $2\pi/N$. However, the standard deviation of $\hat{\omega}$ estimates obtained from (3.5) are $O(1/L\sqrt{N})$, when $L < N^{1/2}$ which asymptotically is too large for use with the Gauss-Newton method. Thus, we need to improve the initial frequency estimates before starting the minimization.

It is known [7] that for $N \to \infty$ the ML estimates of $\{\omega_i\}$ are given by the maxima of the periodogram. Thererfore, one method for improving initial frequency estimates is to search in some small interval about each $\hat{\omega}_i$, say $[\hat{\omega}_i - \varepsilon, \hat{\omega}_i + \varepsilon]$ for the maximum of the periodogram. Specifically, the following method can be used.

Choose appropriate values for $\Delta\omega$ and ℓ_{max}.

For each $i=1,2,\ldots,m$

1) Compute the periodogram $\alpha_{i\ell}$ of the data at frequencies

$$\bar{\omega}_{i\ell} = \hat{\omega}_i \pm \ell\Delta\omega \qquad \ell = 0,1,\ldots,\ell_{max}$$

using

$$\alpha_{i\ell} = (\beta_{i\ell}^2 + b_{i\ell}^2)^{1/2} ,$$

where $\beta_{i\ell}$ and $b_{i\ell}$ are computed using (3.15) but with $M=N$

2) Choose as the new initial frequency estimate the $\bar{\omega}_{i\ell}$ whose corresponding $\alpha_{i\ell}$ is largest; compute the new initial amplitude and

phase estimates using (3.15).

From the above discussion, $\Delta\omega$ should be chosen less than $2\pi/N$ to ensure that one of the $\tilde{\omega}_{i\ell}$ is in the deep trough of LF; in our simulations we used $\Delta\omega\varepsilon[\frac{2\pi}{3N}, \frac{\pi}{N}]$. Moreover, ℓ_{max} should be chosen so that $Pr[\omega_i \varepsilon[\hat{\omega}_i - \ell_{max}\cdot\Delta\omega, \hat{\omega}_i + \ell_{max}\cdot\Delta\omega]$ is sufficiently large. In our simulations we chose $\ell_{max} = 20$; however, more sophisticated procedures could be used. For example, since the OYW method was used to obtain $\hat{\omega}_i$, ℓ_{max} could be chosen based on the asymptotic probability destribution of the ω_i given in [12]. Finally, one must ensure that the search interval for two adjacent frequencies do not overlap.

6. NUMERICAL EXAMPLES

We present some numerical experiments that indicate the performance of the proposed algorithm. We first consider the poblem of estimating θ from a signal of the form (2.1), where m=2,

$$\alpha_1 = 1.0 \qquad \omega_1 = 0.4\pi \qquad \phi_1 = -0.5$$

$$\alpha_2 = 1.0 \qquad \omega_2 = 0.2\pi \qquad \phi_2 = 1.0$$

In all examples $L = \sqrt{N}$, and M is chosen as in (3.17). The white noise variance λ^2 was varied so that the SNR ranges between 0-20 dB in 2.5 dB increments. (Here, SNR is defined for each signal, i.e., $SNR = \alpha_1^2/2\lambda^2$) . For each SNR, 50 independent data sets were generated, and average sum-squared errors (SSE) of the resulting estimates of $K_N\hat{\theta}$ were computed. The SSE is defined as

$$\frac{1}{K}\sum_{i=1}^{k} \| K_N(\hat{\theta}_i-\theta) \|^2$$

where K is the number of independent estimates obtained (50 in these simulations) and $\hat{\theta}_i$ is the i-th estimate vector.

The SSE of the estimated coefficients for N=500 are shown in Figures 6.1-6.3. In these (and the remaining) figures, initial estimates are those obtained using the methods of section 3. Equation (3.15) was used for estimates in these plots; however, the SSE for estimates obtained using (3.9) are not significantly different (and in particular, no better on the average). From these initial estimates, improved estimates were obtained as outlined in the previous section, then the Gauss-Newton algorithm (equation (4.8)) was used. In equation (4.8), μ_k was at each iteration set to 1; if LF increased, μ was decreased by a factor of 4 until the resulting step was such that LF decreased.

Figures 6.4-6.6 shows the SSE of the initial estimates from the OYW method, the improved estimates using the method of Section 5, and the ML estimates. The number of data points, etc. is the same as for figures 6.1-6.3, and only the first parameters α_1, ϕ_1, and ω , are shown. From Figure 6.6 we see that the method of Section 5 significantly improves the initial frequency estimates, especially for low SNR. Moreover, the iterative ML

method provides significant improvement over the modified initial estimates. Note that although amplitude and phase estimates sometimes become worse after the initial frequency improvement, they become much better after the iterative step. As a side note, Figure 6.6 shows that the iterative ML method perform better than regular FFT-based methods, since the frequency estimates after the improvement of section 5 are at least as good as FFT-based estimates.

From figures 6.1-6.3 it can be seen that the SSE of the ML estimates are very close to the Cramer-Rao bound for SNR's above 0 dB. For the SNR of 0 dB, the high SSE's are caused by convergence to a local minimum of LF in only 2 of the 50 cases; and in these two cases, the $\hat{\omega}_1$ estimates were in error by less than 0.006π. Similar performance is evident in Figures 6.7 and 6.8 for N = 1000 and N = 50 data points, respectively.

From these figures it is apparent that there is a SNR threshold above which the ML estimator gives variances that agree closely with the CR lower bounds. Moreover, this threshold decreases with increasing number of data points. This latter fact is evident from figures 6.3-6.5 where the threshold is 5 dB for N = 50 data points, 2.5 dB for N = 500, and 0 dB (or lower) for N = 1000.

Figure 6.9 shows results for N = 500 data points when the frequency difference between ω_1 and ω_2 varies. Specifically, ω_1 is fixed at 0.4π, and ω_2 is varied from 0.2π to 0.4π. The amplitudes and phases are the same as before, and the SNR is 10 dB. When $\omega_2 = 0.375\pi$, the ML method fails to yield better average results than the initial guess; however, these poor averages are caused by failure of the ML method to improve the estimates in only 5-10 of the 50 cases.

In Figure 6.10 we show average error results for N = 500 data points when the additive noise is colored. The noise used is MA(1):

$$n(t) = [\varepsilon(t) + 0.9\varepsilon(t-1)]/\sqrt{1.81}$$

Note that n(t) has the same total power as $\varepsilon(t)$ does. It can be seen in Figure 6.10 that the ML method provides signficant improvement over Yule-

Walker estimates even for colored noise. In fact, the errors in the colored noise case are lower than when white noise was used. This is presumably due to a lower asymptotic CR lower bound for the colored noise case. The Yule-Walker method does not give consistent estimates in this case, because the first row of equation (3.5) should not be used (the data can be modeled as a limiting ARMA(4,5) process, in which case (3.5) holds only for $k > 2$). However, for large L, the effect of the first equation is small, and "reasonable" estimates could still result (as is seen in Figure 6.7). We note also that for colored noise, the proposed method is not a maximum likelihood estimate, but it is still an output error method.

The CPU time needed to obtain the ML estimates was about 10 times that of the time needed to obtain the initial estimates. (The initial estimates required .08-.35 seconds and the ML estimates .70-4.0 seconds on a VAX 11-750 as N ranged from 100-1000). About one-half of the CPU time was spent obtaining improved initial guesses via (5.4), and the other half was spent on the actual function minimization. The minimization procedure rarely required more than 3-4 iterations to achieve a tolerance of 10^{-4} to 10^{-6} (where no element of θ changed more than "tolerance" in one iteration).

As a final note, the recursive computation of $\sin\omega t$ or $\cos\omega t$ using (3.11) required only about 1/6 the CPU time of direct computation. The error between the recursively and directly computed values remained below 10^{-3} for $N < 1000$ (using single precision arithmetic); a typical plot of the error is shown in Figure 6.11.

7. CONCLUSIONS

We derived a (simplified) Gauss-Newton algorithm for estimating the parameters of sinusoidal signals in noise. The algorithm is based on maximization of the likelihood function and is initialized by a set of preliminary estimates obtained via the overdetermined Yule-Walker method. The asymptotic properties of the proposed techniques are discussed and it is shown that the parameter estimates are consistent and asymptotically efficient for the Gaussian case. In the non-Gaussian case the estimator provides a minimum-variance solution within a large class of estimators based on second order statistics.

The performance of the proposed technique and its capability for resolving closely-spaced sinusoids were studied by Monte-Carlo simulations. It was shown that the Gauss-Newton procedure performs better than the overdetermined Yule-Walker method.

APPENDIX A

CRAMÉR-RAO LOWER BOUNDS

The estimation problem formulated in section 2 falls into the class of nonlinear regression problems. The CRLB, say P^N_{CR}, for any unbiased estimator of θ and λ^2 can be easily derived [7]. In this appendix we will be interested in the asymptotic CRLB: P^∞_{CR}. The reason for this interest is threefold:

(i) P^∞_{CR} has a much simpler expression than P^N_{CR} and is, therefore, much easier to compute. Yet P^∞_{CR} is a good approximation of P^N_{CR} whenever

$$\inf_{i \neq j} |\omega_i - \omega_j| > \frac{2\pi}{N} \tag{A.1}$$

This will become apparent in the following, where it will be shown that the smaller the minimum separation in frequency $\inf|\omega_i - \omega_j|$, the slower is the convergence of P^N_{CR} to P^∞_{CR}. It is worth noting that a main conclusion of the study of P^N_{CR} in [7] was that P^N_{CR} increases rapidly as the minimum frequency separation goes below the critical value $2\pi/N$. In such a case P^N_{CR} is much larger than P^∞_{CR}.

(ii) P^N_{CR} can be attained only under certain restrictive conditions [10] which apparently are not satisfied for the problem under study. On the other hand, P^∞_{CR} is attained in the limit (as $N \to \infty$) by the covariance matrix of the ML estimate; see theorem 4.1. Furthermore, for other estimation methods (such as the OYW method) only asymptotic results are available. Thus, it is P^∞_{CR} which is of interest in any analytical study comparing the performance of the ML method with that of other estimation methods.

(iii) The expression of P^∞_{CR} is useful in the derivation of the simplified ML Gauss-Newton algorithm in section 4. Note that an expression for P^∞_{CR} does not seem to be available in the literature, except for the special case of $m = 1$; see [9] and its references.

For the estimation problem under discussion, the log-likelihood function

is given by

$$L(\theta,\lambda^2) = -\frac{N}{2}\ln(2\pi) - \frac{N}{2}\ln\lambda^2 - \frac{1}{2\lambda^2}\sum_{t=1}^{N}\varepsilon^2(t) , \qquad (A.2)$$

where

$$\varepsilon(t) = y(t) - \sum_{i=1}^{m}\alpha_i \sin(\omega_i t + \phi_i) . \qquad (A.3)$$

The CRLB,

$$P_{CR}^N = \left[E\left\{\begin{bmatrix}\frac{\partial L(\theta,\lambda^2)}{\partial\theta} \\ \frac{\partial L(\theta,\lambda^2)}{\partial\lambda^2}\end{bmatrix}\cdot\left[\left(\frac{\partial L(\theta,\lambda^2)}{\partial\theta}\right)^T \frac{\partial L(\theta,\lambda^2)}{\partial\lambda^2}\right]\right\}\right]^{-1} , \qquad (A.4)$$

can be evaluated by straightforward calculations:

$$\frac{\partial L}{\partial\theta} = -\frac{1}{\lambda^2}\sum_{t=1}^{N}\varepsilon(t)\varepsilon_\theta(t)$$

$$\frac{\partial L}{\partial\lambda^2} = -\frac{N}{2\lambda^2} + \frac{1}{2\lambda^4}\sum_{t=1}^{N}\varepsilon^2(t)$$

$$E\left[\frac{\partial L}{\partial\theta}\cdot\left(\frac{\partial L}{\partial\theta}\right)^T\right] = \frac{1}{\lambda^2}\sum_{t=1}^{N}\varepsilon_\theta(t)\ \varepsilon_\theta^T(t)$$

$$E\left[\frac{\partial L}{\partial\theta}\cdot\frac{\partial L}{\partial\lambda^2}\right] = \frac{N}{2\lambda^4}\sum_{t=1}^{N}\varepsilon_\theta(t)E\{\varepsilon(t)\} -$$

$$-\frac{1}{2\lambda^6}\sum_{t=1}^{N}\sum_{s=1}^{N}\varepsilon_\theta(t)E\{\varepsilon(t)\varepsilon^2(s)\} = 0 \qquad (A.5)$$

$$E\left[\frac{\partial L}{\partial\lambda^2}\right]^2 = \frac{N^2}{4\lambda^4} + \frac{1}{4\lambda^8}\sum_{t=1}^{N}\sum_{s=1}^{N}E\ \varepsilon^2(t)\varepsilon^2(s) - \frac{N^2}{2\lambda^4} =$$

$$= -\frac{N^2}{4\lambda^4} + \left[\frac{3N}{4\lambda^4} + \frac{N(N-1)}{4\lambda^4}\right] = \frac{N}{2\lambda^4} . \qquad (A.6)$$

In (A.5) and (A.6) we used the assumption that $\varepsilon(t)$ is white Gaussian noise. It follows that

$$P_{CR}^{N} = \begin{bmatrix} P_{CR}^{\theta,N} & 0 \\ 0 & \frac{2\lambda^4}{N} \end{bmatrix} , \tag{A.7}$$

where

$$P_{CR}^{\theta,N} = \lambda^2 \left[\sum_{t=1}^{N} \varepsilon_\theta(t)\varepsilon_\theta^T(t) \right]^{-1} , \tag{A.8}$$

and where the derivates of $\varepsilon(t)$ with respect to the parameters $\{\alpha_i, \phi_i, \omega_i\}$ are given by (4.3). The expression (A.8) for $P_{CR}^{\theta,N}$ appears, for example, in [7]. However, the calculations necessary to show that P_{CR}^{N} has the block-diagonal form of (A.7), which in turn implies that $P_{CR}^{\theta,N}$ is given by (4.8) were not included there.

In the following we will study the limit of $P_{CR}^{\theta,N}$ as $N \to \infty$. The following results will be useful for this study.

<u>Lemma A.1.</u>

For $\omega \in [0, 2\pi)$,

$$\frac{1}{N}\sum_{t=1}^{N} \cos(\omega t + \phi) = \begin{cases} \cos\phi & \text{for } \omega = 0 \\ \frac{1}{N} \dfrac{\sin(\frac{N\omega}{2})\cos(\frac{N+1}{2}\omega+\phi)}{\sin(\frac{\omega}{2})} & \text{for } \omega \neq 0 \end{cases} \tag{A.9}$$

<u>Proof</u>: [17].

<u>Corrolary</u>.

For $\omega\varepsilon[0, 2\pi)$,

$$\lim_{N\to\infty} \frac{1}{N^{k+1}} \sum_{t=1}^{N} t^k \cos(\omega t+\phi) = \begin{cases} \frac{1}{k+1}\cos\phi & \omega = 0 \\ 0 & \omega \neq 0 \end{cases}, \quad k > 0 \tag{A.10}$$

Proof: For k=0 the limit follows immediately from (A.9). For k > 0 the limits follow from relations similar to (A.9) obtained by differentiation of (A.9) with respect to ω.

Let us denote

$$\overline{P}^{\theta,N}_{CR} = K_N \, P^{\theta,N}_{CR} \, K_N , \tag{A.11}$$

where K_N is given by (4.4). Clearly $\overline{P}^{\theta,N}_{C_R}$ is the CRLB on the covariance matrix of the following normalized estimation error vector

$$\begin{bmatrix} \sqrt{N}(\hat{\alpha} - \alpha) \\ \text{- - - - - -} \\ \sqrt{N}(\hat{\phi} - \phi) \\ \text{- - - - - -} \\ N\sqrt{N}(\hat{\omega} - \omega) \end{bmatrix} , \tag{A.12}$$

where $\alpha = [\alpha_1, \ldots, \alpha_m]^T$, and $\hat{\alpha}$ is any unbiased estimator of α. ϕ, $\hat{\phi}$ and $\hat{\omega}$, ω are similarly defined. In the following we will show that

$$P^{\theta}_{CR} \triangleq \lim_{N\to\infty} \overline{P}^{\theta,N}_{CR} \tag{A.13}$$

exists and has a simple expression.

By making repeated use of Lemma A.1 and its corrolary we can write, see (A.9)

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} \frac{\partial\varepsilon(t)}{\partial\alpha_i} \cdot \frac{\partial\varepsilon(t)}{\partial\alpha_j} = \frac{1}{2} \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} \{\cos[(\omega_i-\omega_j)t + \phi_i - \phi_j]$$

$$- \cos[(\omega_i+\omega_j)t + \phi_i + \phi_j]\} = \frac{1}{2}\delta_{i,j}$$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} \frac{\partial\varepsilon(t)}{\partial\alpha_i} \cdot \frac{\partial\varepsilon(t)}{\partial\phi_j} = \frac{\alpha_j}{2} \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} \{\sin[(\omega_i+\omega_j)\, t + \phi_i + \phi_j]$$

$$+ \sin[(\omega_i-\omega_j)t + \phi_i - \phi_j]\} = 0$$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} \frac{\partial\varepsilon(t)}{\partial\phi_i} \cdot \frac{\partial\varepsilon(t)}{\partial\phi_j} = \frac{\alpha_i\alpha_j}{2} \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} \{\cos[(\omega_i-\omega_j)t +$$

$$+ \phi_i - \phi_j] + \cos[(\omega_i+\omega_j)t + \phi_i + \phi_j]\} = \frac{\alpha_i\alpha_j}{2} \delta_{i,j} \tag{A.14}$$

$$\lim_{N\to\infty} \frac{1}{N^2} \sum_{t=1}^{N} \frac{\partial\varepsilon(t)}{\partial\alpha_i} \cdot \frac{\partial\varepsilon(t)}{\partial\omega_j} = \frac{\alpha_j}{2} \lim_{N\to\infty} \frac{1}{N^2} \sum_{t=1}^{N} t\{\sin[(\omega_i+\omega_j)t + \phi_i + \phi_j]$$

$$+ \sin[(\omega_i-\omega_j)t + \phi_i - \phi_j]\} = 0$$

$$\lim_{N\to\infty} \frac{1}{N^2} \sum_{t=1}^{N} \frac{\partial\varepsilon(t)}{\partial\phi_i} \cdot \frac{\partial\varepsilon(t)}{\partial\omega_j} = \frac{\alpha_i\alpha_j}{2} \lim_{N\to\infty} \frac{1}{N^2} \sum_{t=1}^{N} t\{\cos[(\omega_i-\omega_j)t + \phi_i - \phi_j]$$

$$+ \cos[(\omega_i+\omega_j)t + \phi_i + \phi_j]\} = \frac{\alpha_i\alpha_j}{4} \delta_{i,j}$$

$$\lim_{N\to\infty} \frac{1}{N^3} \sum_{t=1}^{N} \frac{\partial\varepsilon(t)}{\partial\omega_i} \cdot \frac{\partial\varepsilon(t)}{\partial\omega_j} = \frac{\alpha_i\alpha_j}{2} \lim_{N\to\infty} \frac{1}{N^3} \sum_{t=1}^{N} t^2\{\cos[(\omega_i-\omega_j)t + \phi_i - \phi_j] +$$

$$+ \cos[(\omega_i+\omega_j)t + \phi_i + \phi_j]\} = \frac{\alpha_i\alpha_j}{6} \delta_{i,j}$$

where $\delta_{i,j}$ denotes Dirac's delta (3.3b). Therefore,

$$P_{CR}^{\theta} = \lambda^2 \left[\begin{array}{c|c|c} \frac{1}{2} I_m & 0 & 0 \\ \hline 0 & \begin{matrix} \frac{\alpha_1^2}{2} & & 0 \\ & \ddots & \\ 0 & & \frac{\alpha_m^2}{2} \end{matrix} & \begin{matrix} \frac{\alpha_1^2}{4} & & 0 \\ & \ddots & \\ 0 & & \frac{\alpha_m^2}{4} \end{matrix} \\ \hline 0 & \begin{matrix} \frac{\alpha_1^2}{4} & & 0 \\ & \ddots & \\ 0 & & \frac{\alpha_m^2}{4} \end{matrix} & \begin{matrix} \frac{\alpha_1^2}{6} & & 0 \\ & \ddots & \\ 0 & & \frac{\alpha_m^2}{6} \end{matrix} \end{array}\right] , \tag{A.15}$$

which after some straightforward calculations gives

$$
P_{CR}^{\theta} = 2\lambda^2 \left[\begin{array}{c|c|c}
I_m & 0 & 0 \\
\hline
0 & \begin{matrix} \frac{4}{\alpha_1^2} & & 0 \\ & \ddots & \\ 0 & & \frac{4}{\alpha_m^2} \end{matrix} & \begin{matrix} -\frac{6}{\alpha_1^2} & & 0 \\ & \ddots & \\ 0 & & -\frac{6}{\alpha_m^2} \end{matrix} \\
\hline
0 & \begin{matrix} -\frac{6}{\alpha_1^2} & & 0 \\ & \ddots & \\ 0 & & -\frac{6}{\alpha_m^2} \end{matrix} & \begin{matrix} \frac{12}{\alpha_1^2} & & 0 \\ & \ddots & \\ 0 & & \frac{12}{\alpha_m^2} \end{matrix}
\end{array}\right] . \qquad (A.16)
$$

Note that the bounds for phases and frequencies are proportional to the noise-to-signal ratios corresponding to the frequency in question. However, somewhat contrary to intuition the bound for the amplitudes of the sinusoids is independent of these amplitudes. Note also, the almost diagonal structure of P_{CR}^{θ}. The estimation errors of the phase and frequency of the same sinusoid are asymptotically cross-correlated. All the other estimation errors are asymptotically uncorrelated.

It is also interesting to note that the bounds for $(\hat{\omega}_i - \omega_i)$ are of order $1/N^{3/2}$ (see also [9] and its references). This order of the CRLB is rather unusual for a stationary estimation problem for which the corresponding bounds are in general of the order $1/N^{1/2}$. However, the problem of estimating the parameters of a sinusoidal signal is not a strictly stationary estimation problem: the derivative of $\varepsilon(t)$ with respect to ω_i is clearly a nonstationary signal.

It follows from Lemma A.1 that the smaller the minimum frequency separation $\inf_{i \neq j} |\omega_i - \omega_j|$ the slower is the convergence in (A.13). Consider, for example, (A.9) for ω small but non- zero. Then the left-hand-side of (A.9) will generally be small provided that $N\omega$, rather than N, is large enough, see the right-hand-side of (A.9).

APPENDIX B

Since estimates of ω, ϕ, and α are $O(1/\sqrt{N})$ or $O(1/\sqrt{N^3})$, we restrict attention to the case $\omega \simeq \hat{\omega}$, $\alpha \approx \hat{\alpha}$, and $\phi \approx \hat{\phi}$. From (5.3) the derivative of F with respect to $\hat{\omega}$ is

$$\frac{2F}{2\hat{\omega}} = \frac{1}{N}\sum_{t=1}^{N} t[\alpha\sin(\omega t+\phi) - \hat{\alpha}\sin(\hat{\omega}t+\hat{\phi})][-2\hat{\alpha}\cos(\hat{\omega}t+\hat{\phi})] \tag{B.1a}$$

$$= -\frac{\alpha\hat{\alpha}}{N}\sum_{t=1}^{N} t\sin[(t+\hat{\omega})t+(\phi+\hat{\phi})] + \frac{\hat{a}^2}{N}\sum_{t=1}^{N} t\sin(2\hat{\omega}t+2\hat{\phi}) \tag{B.1b}$$

$$-\frac{\alpha\hat{\alpha}}{N}\sum_{t=1}^{N} t\sin[(\omega-\hat{\omega})t + (\phi-\hat{\phi})]$$

We claim that the zeroes of $2F/2\hat{\omega}$ in the region of interest are nearly equal to those of the third term of (B.1b). To support this, a plot of

$$\frac{1}{N}\sum_{t=1}^{N} t\sin\psi t$$

for N=100 is shown in Figure B.1. It can be seen that for ψ not near zero, this function is near zero. Also near $\psi=0$ the zero crossings have large slopes and are therefore insensitive to small additive disturbances. Defining $\tilde{\omega} = \omega - \hat{\omega}$ and $\tilde{\phi} = \phi - \hat{\phi}$, the third term in (B.1b) can be expressed as

$$-\frac{\alpha\hat{\alpha}}{N}\sum_{t=1}^{N} t\sin(\tilde{\omega}t + \tilde{\phi})$$

$$= -\alpha\hat{\alpha}[\cos\tilde{\phi}\ (\sum_{t=1}^{N} t\sin\tilde{\omega}t) + \sin\tilde{\phi}(\frac{1}{N}\sum_{t=1}^{N} t\cos\tilde{\omega}t)] \tag{B.3}$$

Since $\tilde{\phi}$ is $O(1/\sqrt{N})$, the second term in (B.3) can be neglected. Thus for $\omega \approx \hat{\omega}$, $\alpha \approx \hat{\alpha}$ and $\phi \simeq \hat{\phi}$, the zeroes of $\partial F/\partial\hat{\omega}$ are nearly those of the function

$$\frac{1}{N}\sum_{t=1}^{N} t\sin\tilde{\omega}t \tag{B.4}$$

It is not difficult to see that (B.4) is zero for $\tilde{\omega} = 0$. Morever, for $0 < \tilde{\omega} < \frac{\pi}{N}$, (B.4) is positive (since each element of the sum is positive). For $\tilde{\omega} = \frac{2\pi}{N}$, (B.4) is negative; thus the first positive zero of (B.4) occurs for $\tilde{\omega} \in [\frac{\pi}{2N}, \frac{\pi}{N}]$. Since (B.4) is an even function of $\tilde{\omega}$, and since the

zeroes of (B.4) are approximately equal to the zeroes of $\partial F/\partial\hat{\omega}$, we conclude that the width of the main valley of $F(\hat{\omega})$ is approximately in the range

$$(\omega-\hat{\omega}) \in \left[2 \cdot \frac{\pi}{2N} , \; 2 \cdot \frac{\pi}{N}\right] = \left[\frac{\pi}{N} , \; \frac{2\pi}{N}\right] .$$

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Figure Titles

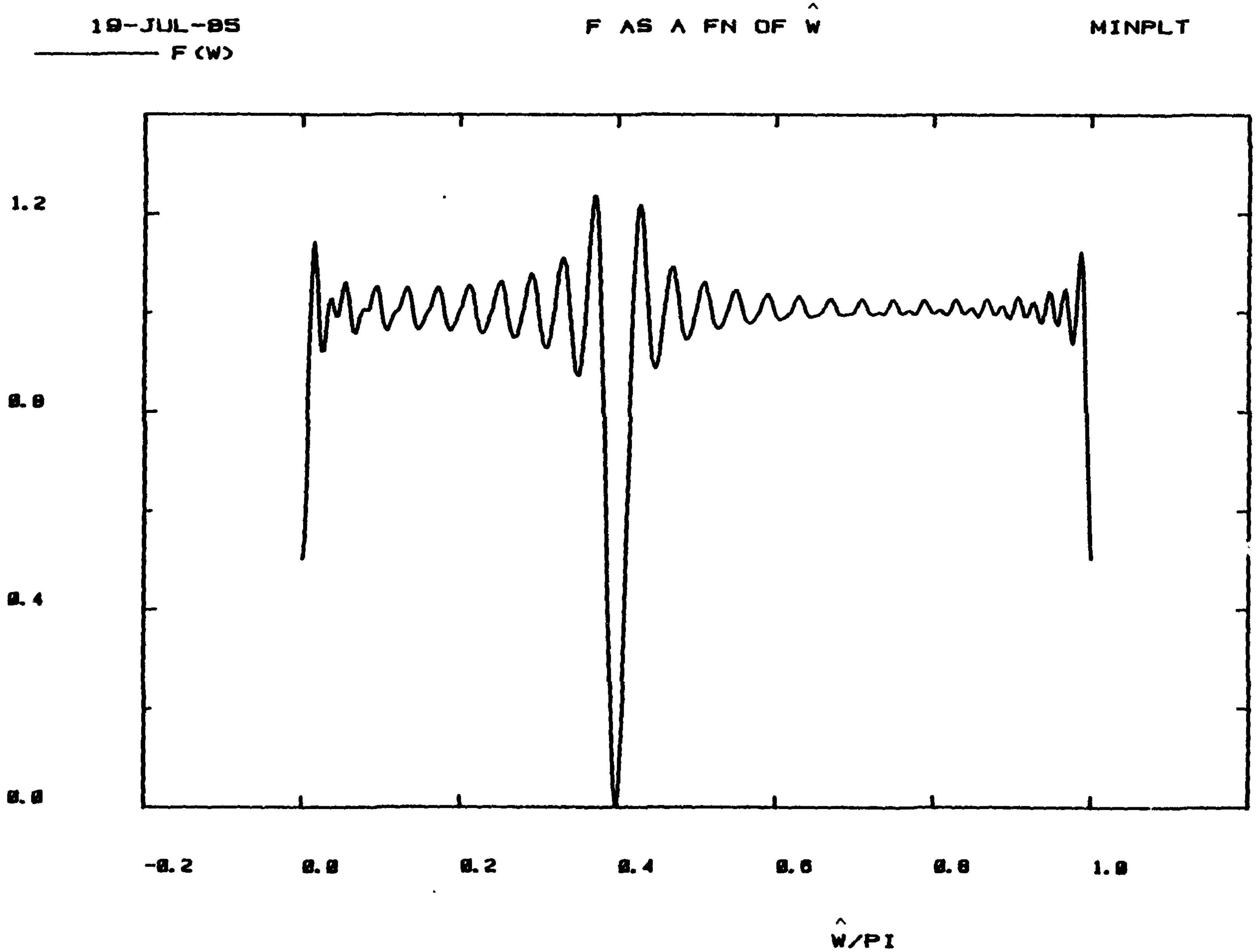


Figure 5.1

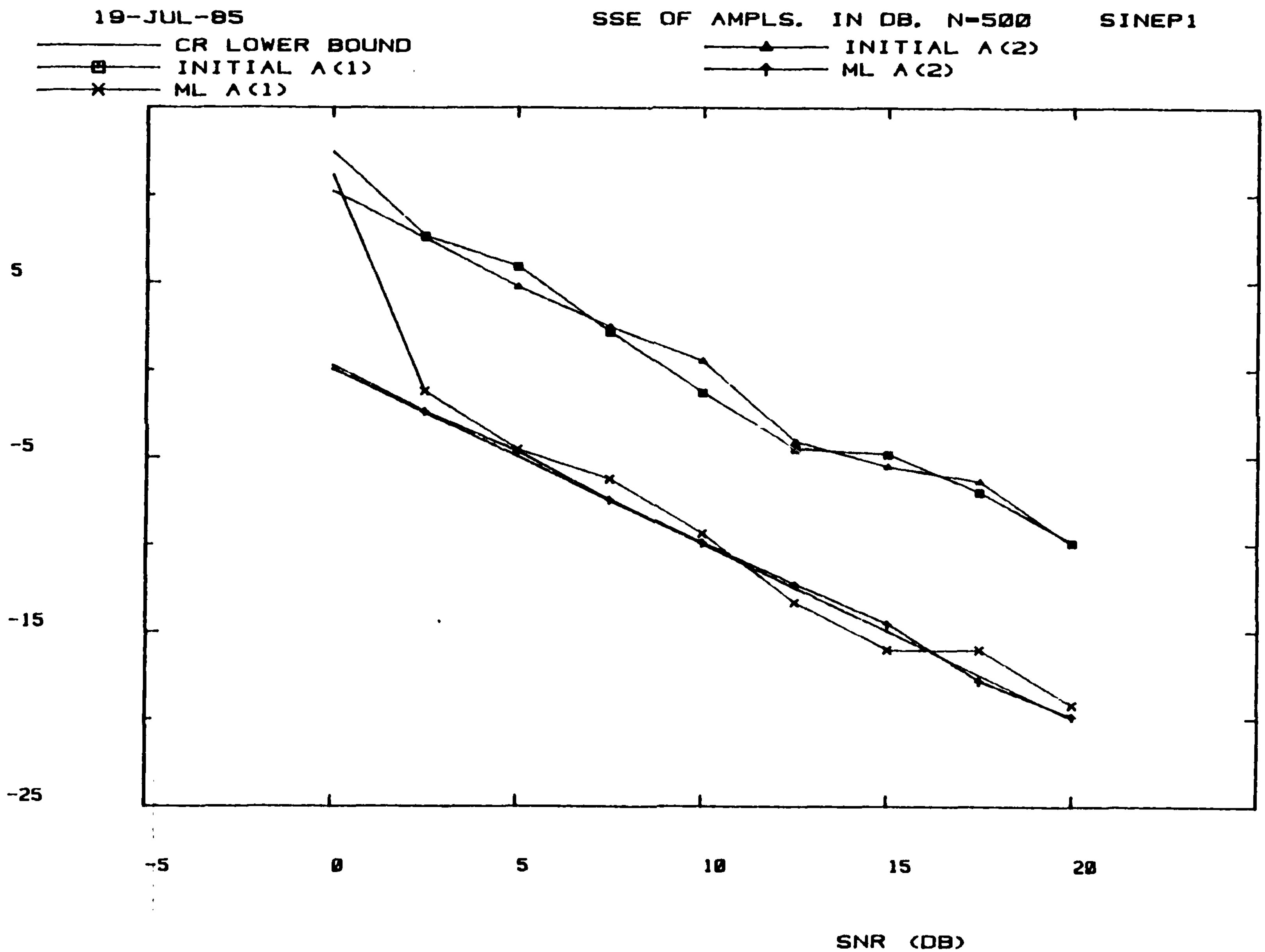
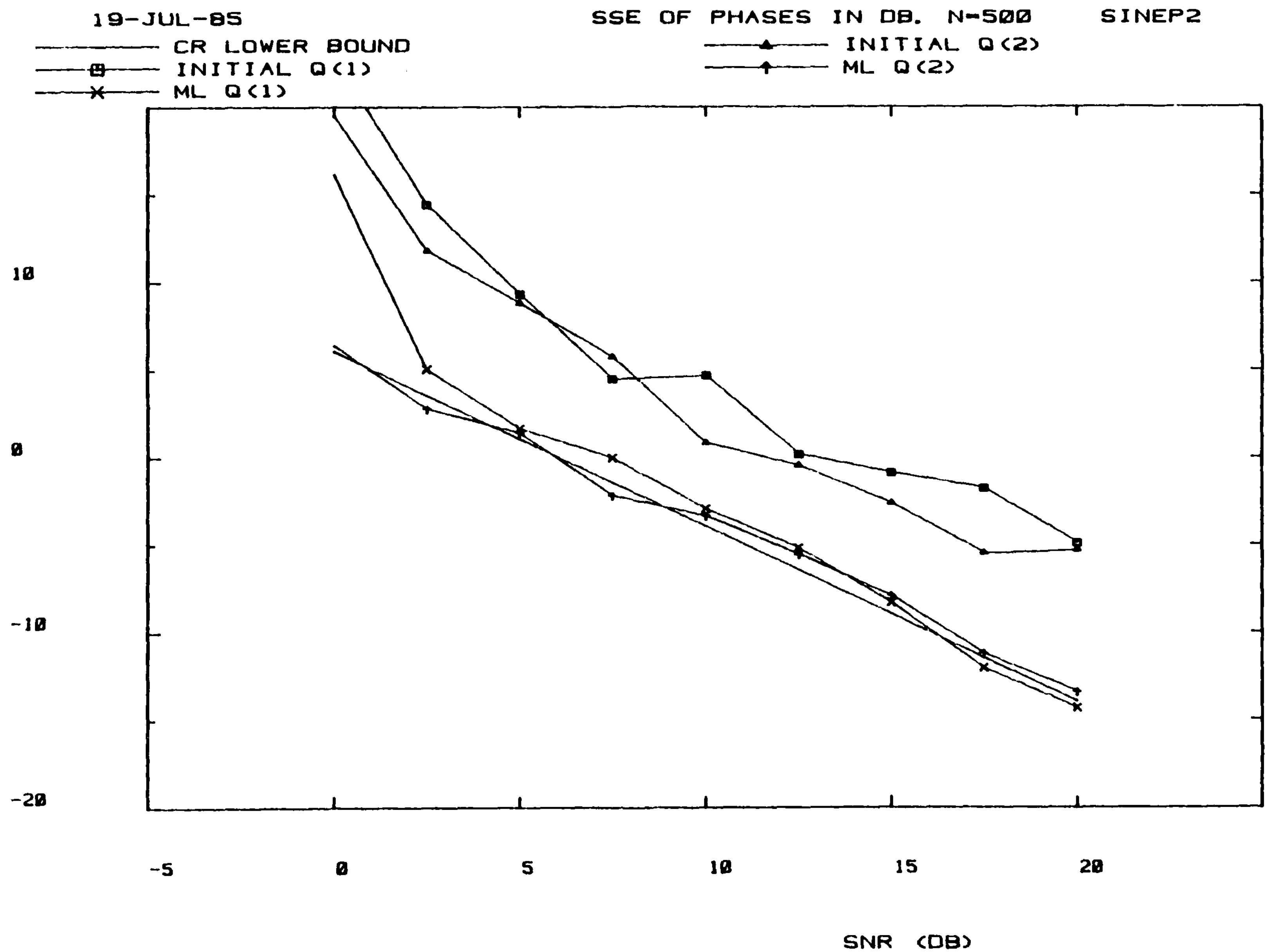

19-JUL-85
SSE OF AMPLS. IN DB. N=500
SINEP1
CR LOWER BOUND
INITIAL A(1)
ML A(1)
INITIAL A(2)
ML A(2)
5
-5
-15
-25
-5
0
5
10
15
20
SNR (DB)

Figure 6.1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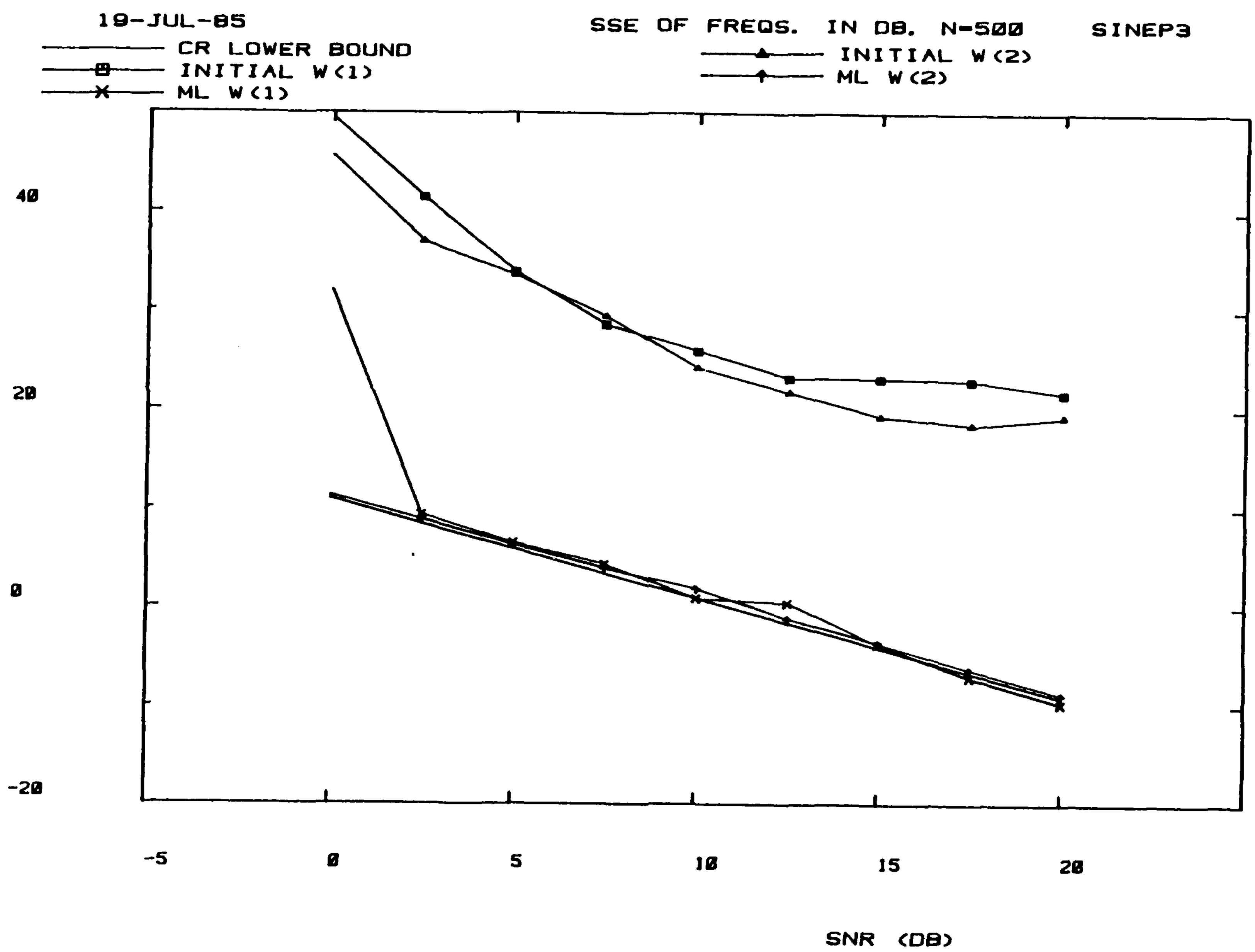
SSE OF PHASES IN DB. N=500
SINEP2
CR LOWER BOUND
INITIAL Q(1)
ML Q(1)
INITIAL Q(2)
ML Q(2)
10
0
-10
-20
-5
0
5
10
15
20
SNR (DB)

Figure 6.2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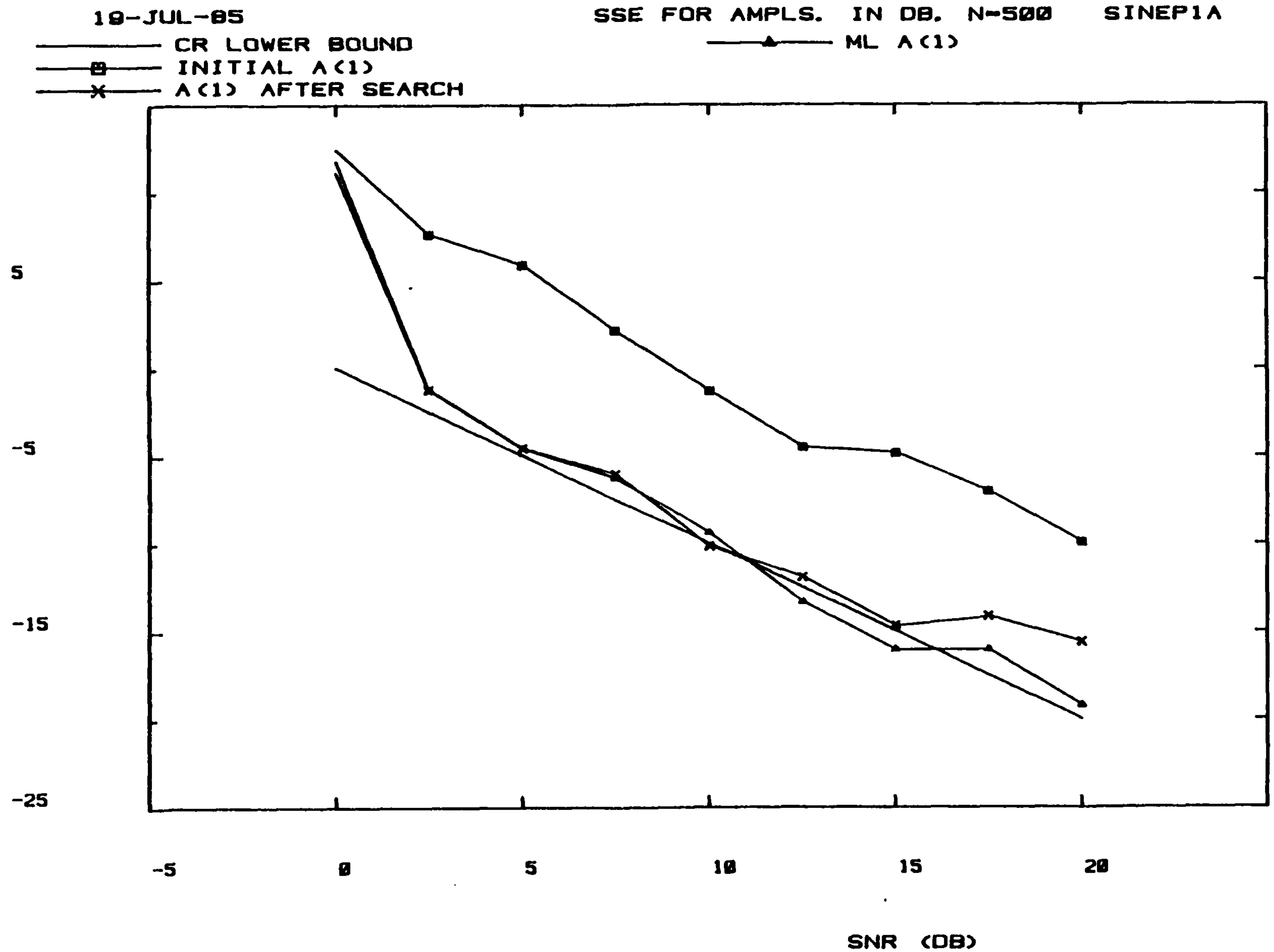
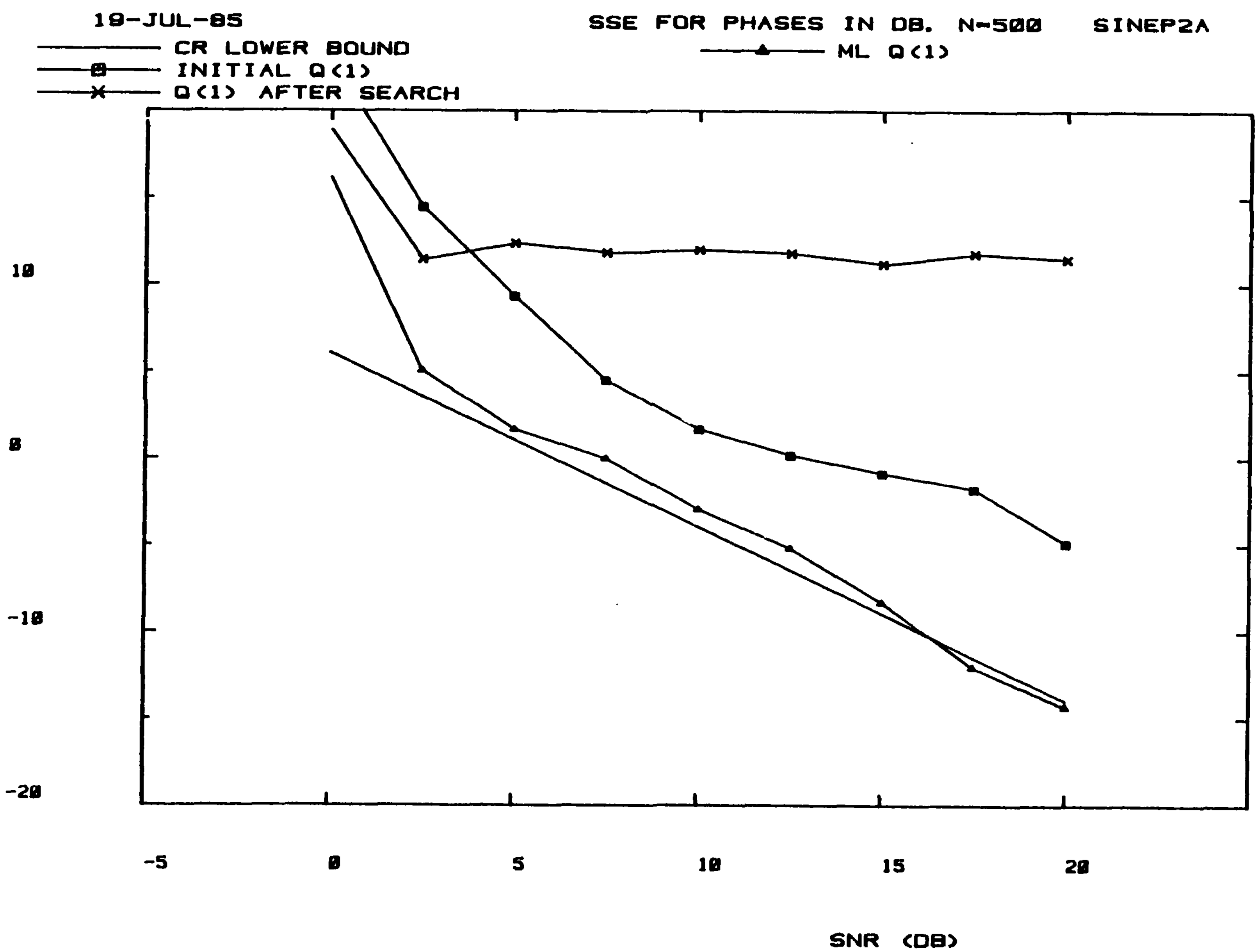
SSE OF FREQS. IN DB. N=500
SINEP3
CR LOWER BOUND
INITIAL W(1)
ML W(1)
INITIAL W(2)
ML W(2)
40
20
0
-20
-5
0
5
10
15
20
SNR (DB)

Figure 6.3



19-JUL-85
SSE FOR AMPLS. IN DB. N=500 SINEP1A
CR LOWER BOUND
INITIAL A(1)
A(1) AFTER SEARCH
ML A(1)
5
-5
-15
-25
-5
0
5
10
15
20
SNR (DB)

Figure 6.4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SSE FOR PHASES IN DB. N=500 SINEP2A
CR LOWER BOUND
INITIAL Q(1)
Q(1) AFTER SEARCH
ML Q(1)
10
0
-10
-20
-5
0
5
10
15
20
SNR (DB)

Figure 6.5

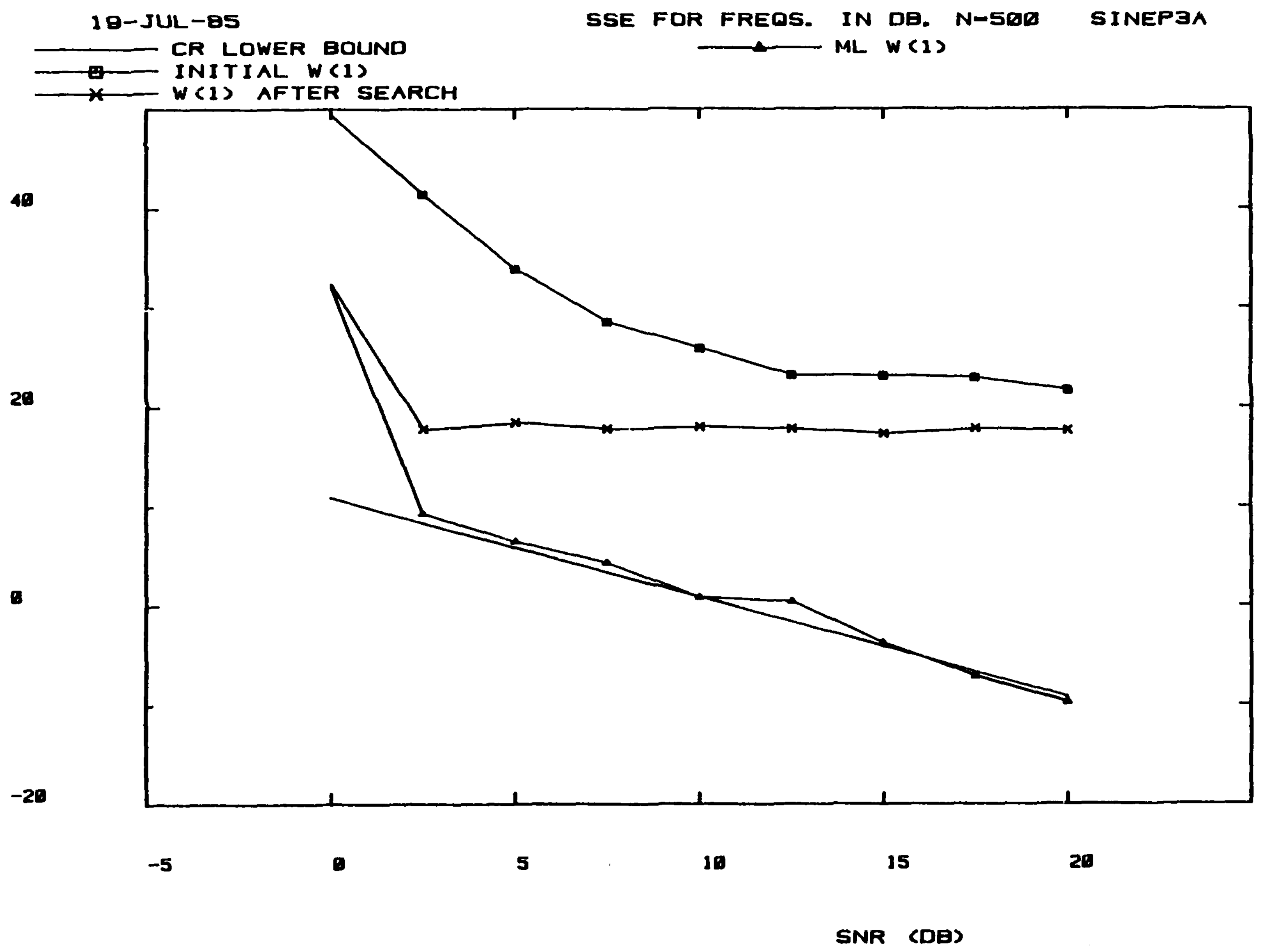
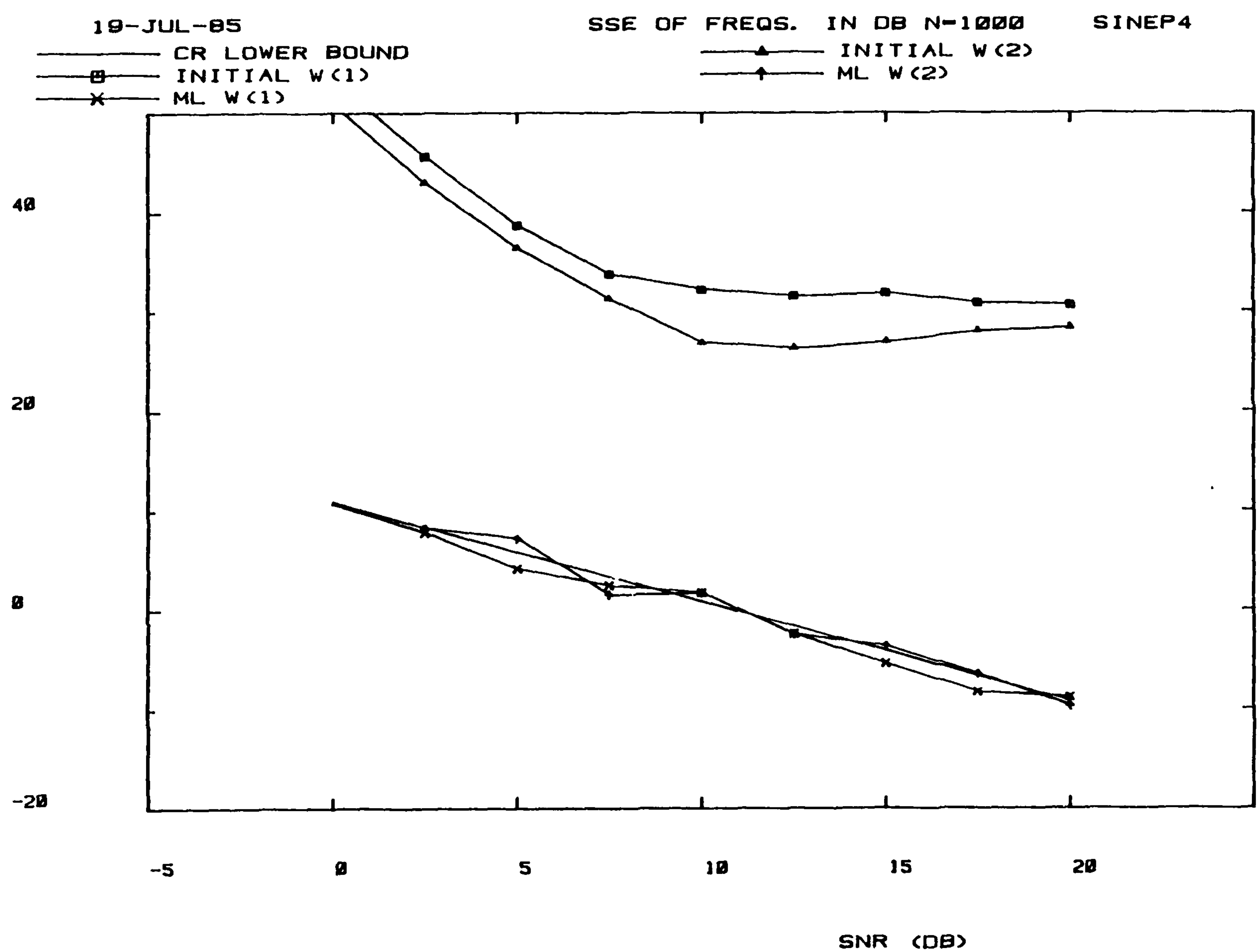

19-JUL-85
SSE FOR FREQS. IN DB. N=500
SINEP3A
CR LOWER BOUND
INITIAL W(1)
W(1) AFTER SEARCH
ML W(1)
40
20
0
-20
-5
0
5
10
15
20
SNR (DB)

Figure 6.6



19-JUL-85
SSE OF FREQS. IN DB N=1000
SINEP4
CR LOWER BOUND
INITIAL W(1)
ML W(1)
INITIAL W(2)
ML W(2)
40
20
0
-20
-5
0
5
10
15
20
SNR (DB)

Figure 6.7

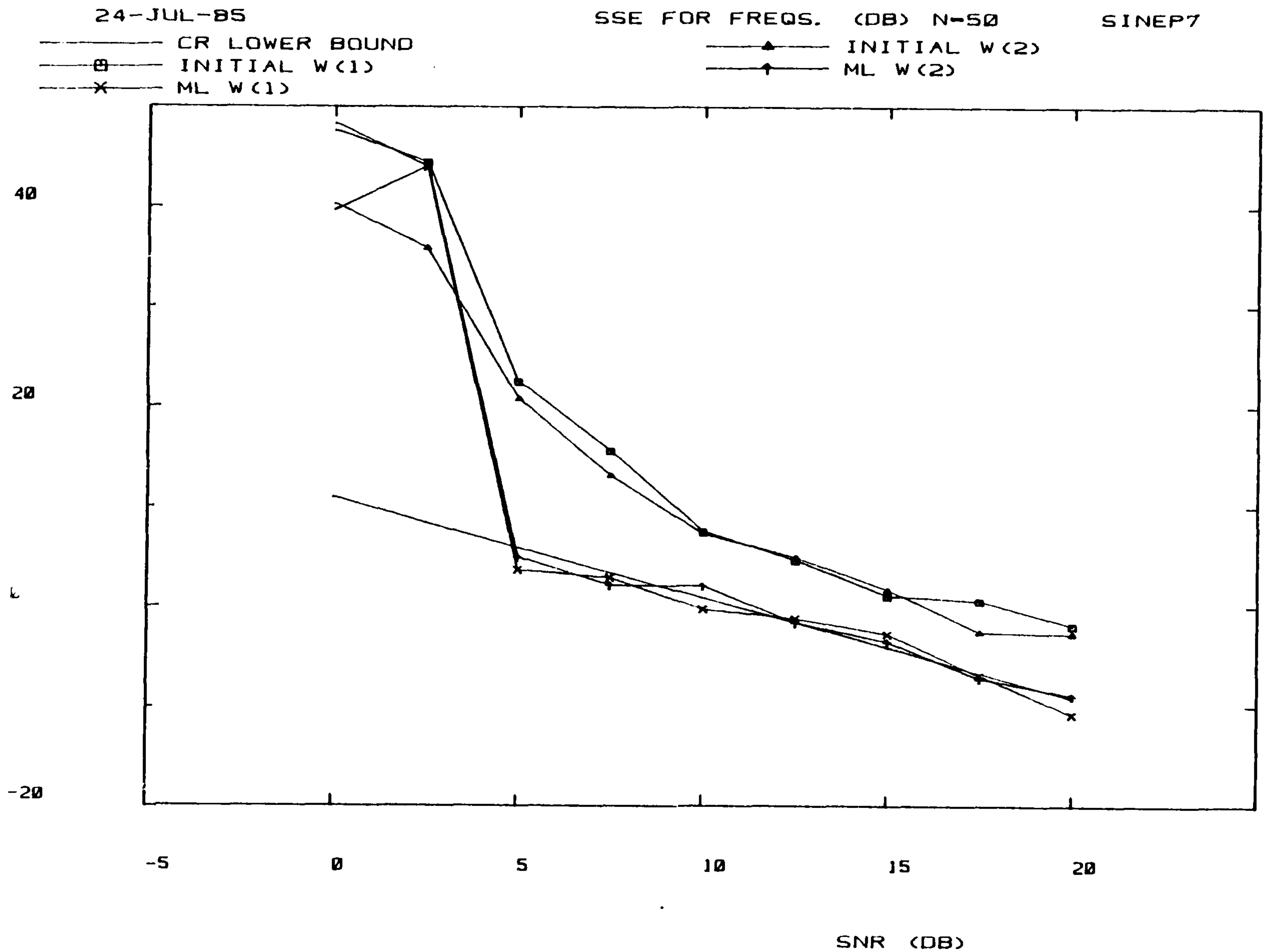
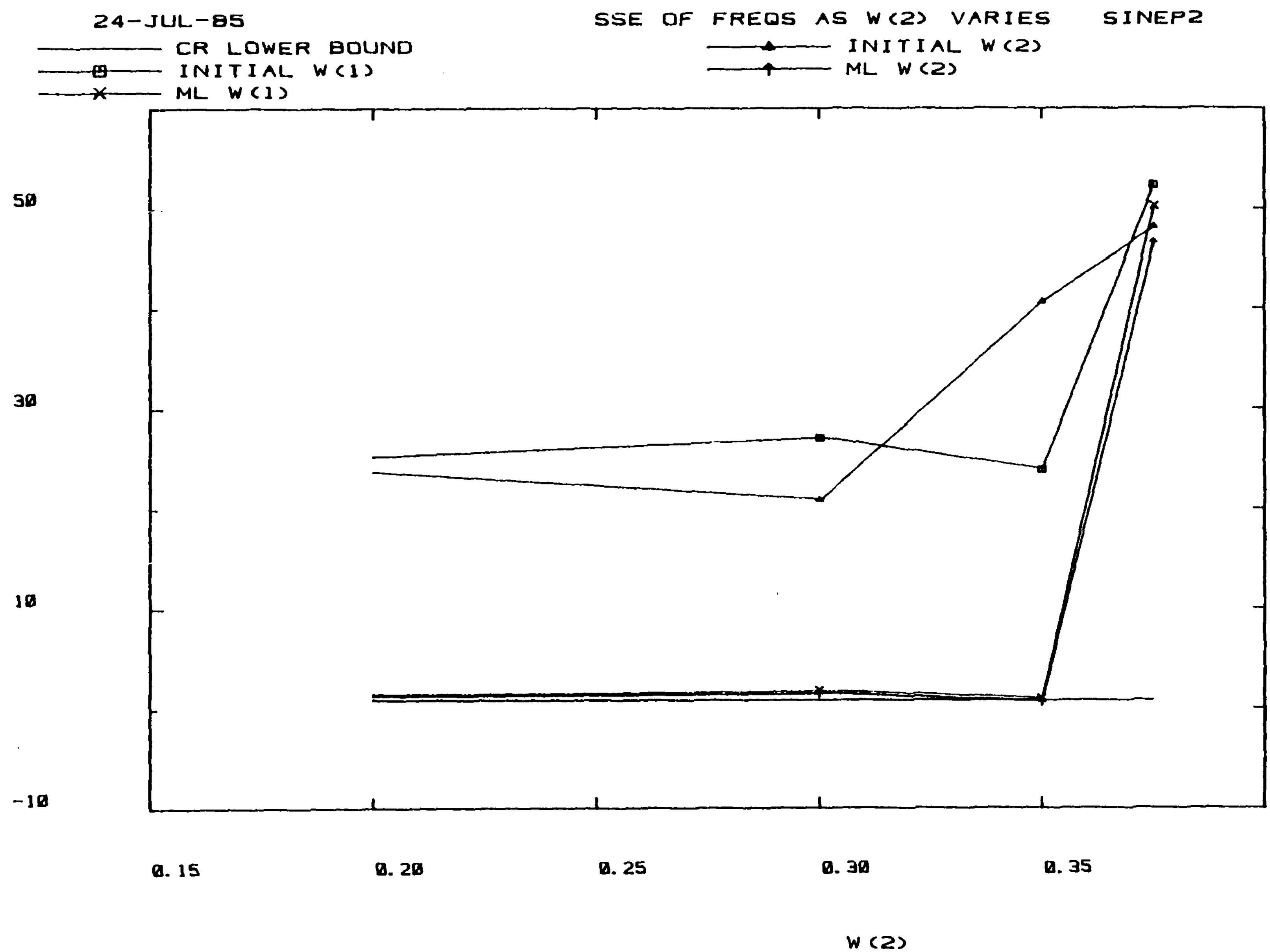

24-JUL-85
SSE FOR FREQS. (DB) N=50
SINEP7
CR LOWER BOUND
INITIAL W(1)
ML W(1)
INITIAL W(2)
ML W(2)
40
20
-20
-5
0
5
10
15
20
SNR (DB)

Figure 6.8



24-JUL-85
SSE OF FREQS AS W(2) VARIES SINEP2
CR LOWER BOUND
INITIAL W(1)
ML W(1)
INITIAL W(2)
ML W(2)
50
30
10
-10
0.15
0.20
0.25
0.30
0.35
W(2)

Figure 6.9

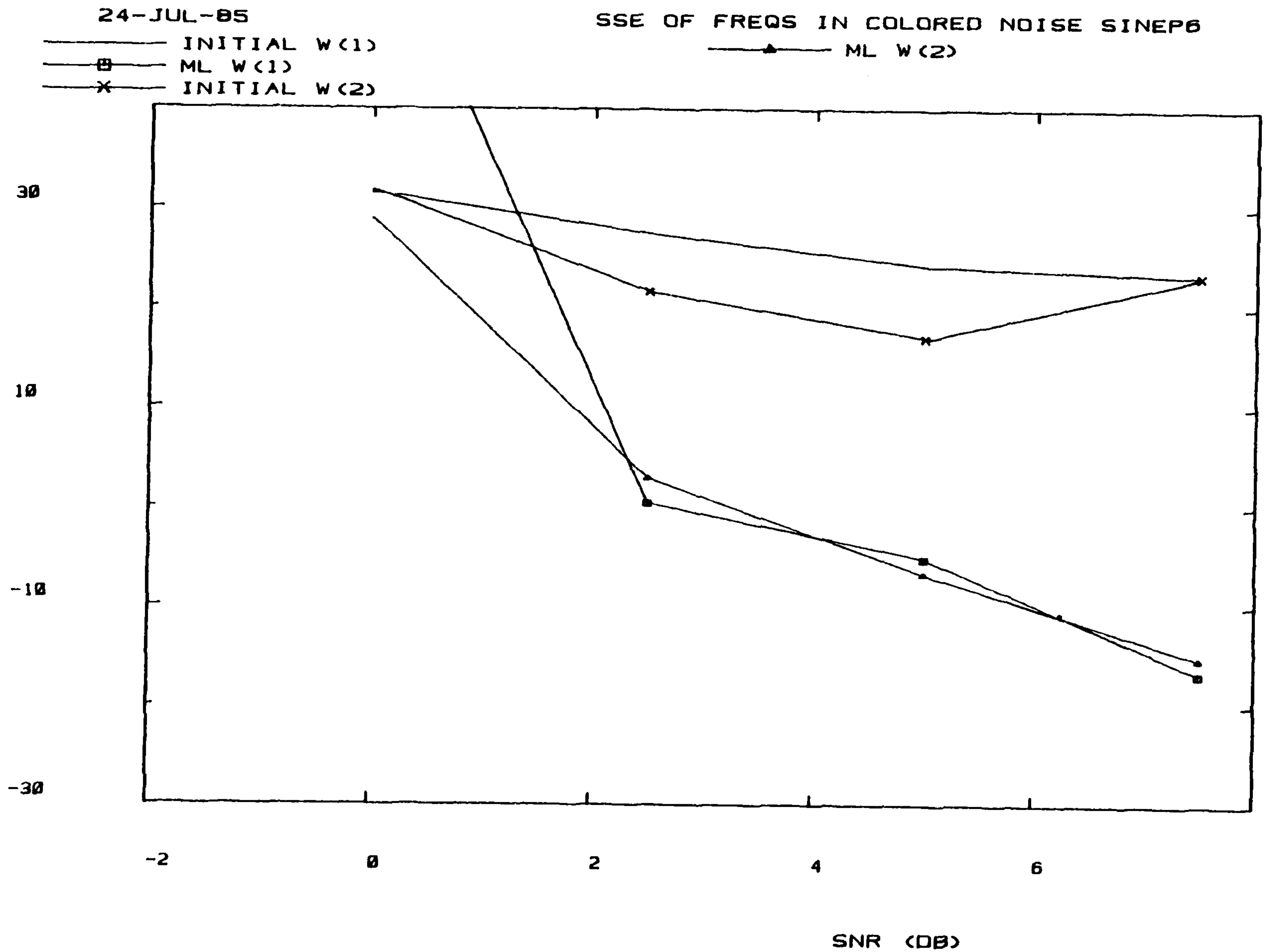
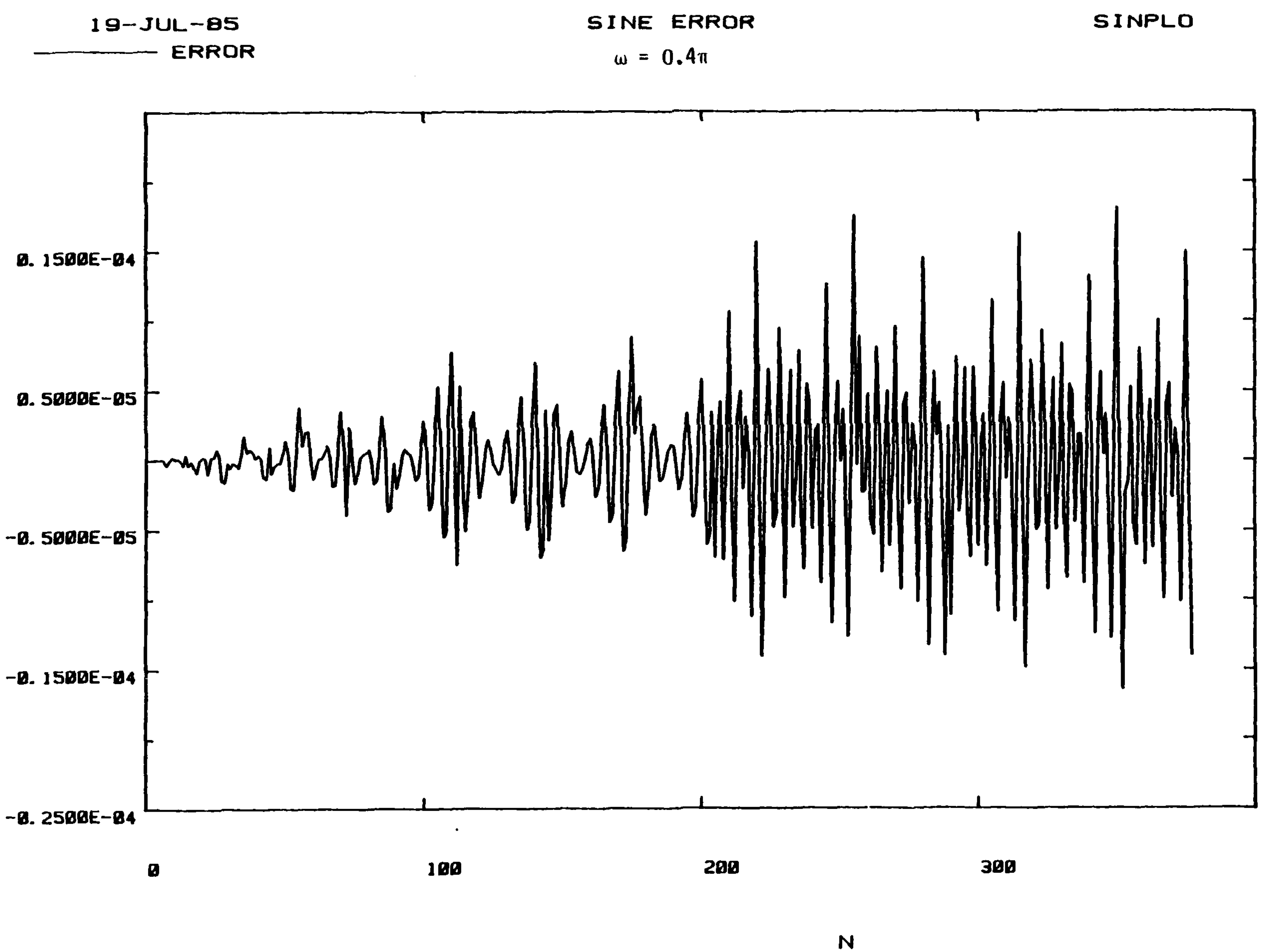

24-JUL-85
SSE OF FREQS IN COLORED NOISE SINEP6
INITIAL W(1)
ML W(1)
INITIAL W(2)
ML W(2)
30
10
-10
-30
-2
0
2
4
6
SNR (DB)

Figure 6.10



19-JUL-85
ERROR
SINE ERROR
ω = 0.4π
SINPLO
0.1500E-04
0.5000E-05
-0.5000E-05
-0.1500E-04
-0.2500E-04
0
100
200
300
N

Figure 6.11

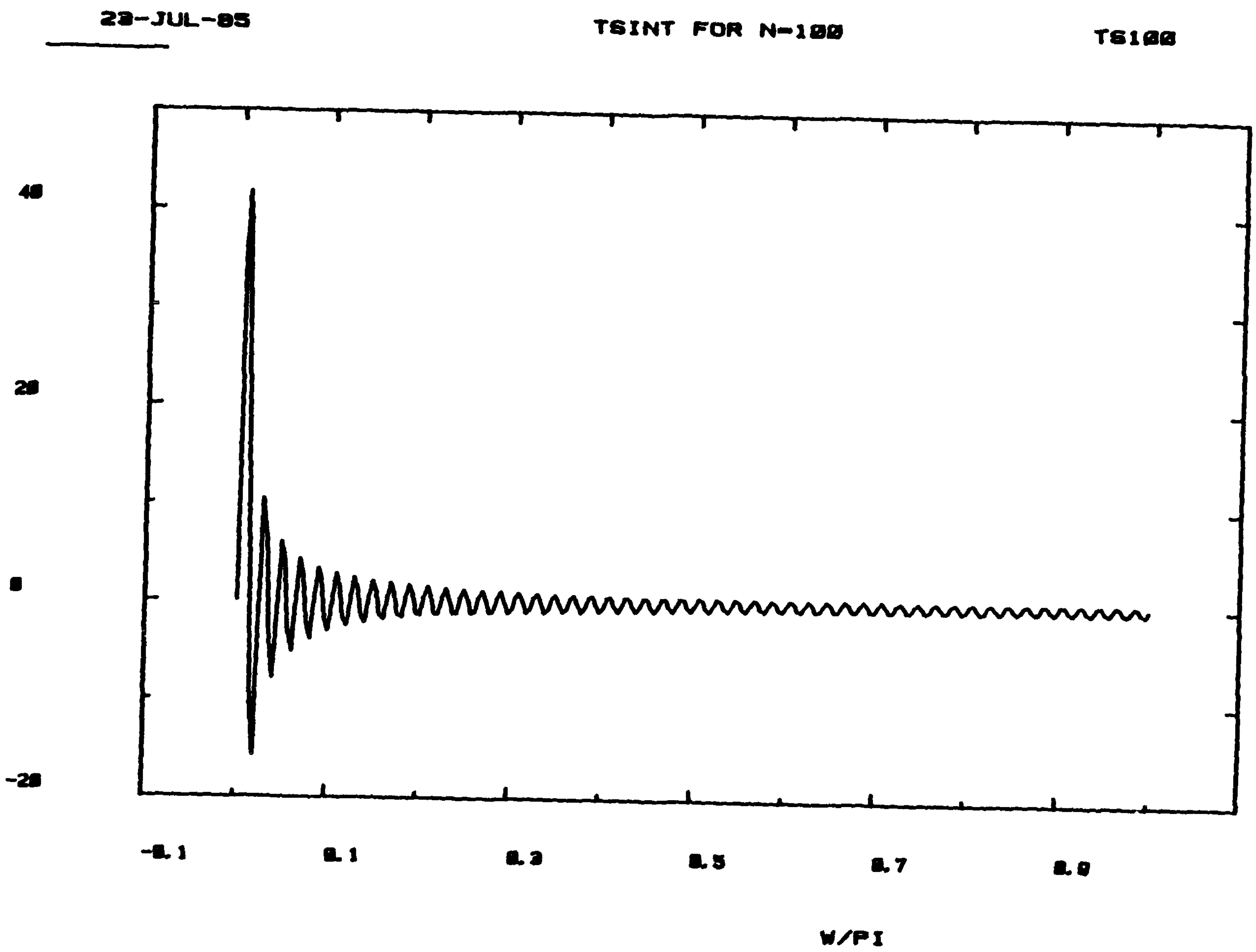


Figure B.1

APPENDIX I

ADAPTIVE DETECTION OF TRANSIENT SIGNALS

6/6/85

ADAPTIVE DETECTION OF TRANSIENT SIGNALS

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ABSTRACT

The paper discusses the problem of detecting transient signals of unknown waveforms in white Gaussian noise. The signals are modeled as impulse responses of rational transfer functions with unknown parameters. A generalized likelihood ratio test (GLRT) is proposed and its statistical properties are analyzed for both known and unknown noise variances. The GLRT involves constrained maximum likelihood estimation of the signal parameters. The performance of the GLRT is compared to that of an optimal matched filter and an energy detector, for some test cases. Also, the theoretical distributions of the likelihood ratios under H_0 and H_1 are compared to experimental distributions obtained by Monte-Carlo simulations.

This work was supported by the Army Research Office Under Contract No. DAAG29-83-C-0027.

1. INTRODUCTION

The problem of detecting transient signals with unknown waveforms arises in the areas of seismic signal processing, underwater surveillance, and other engineering applications. Some common approaches to the problem include: (i) Energy detection, i.e., comparison of the total signal energy to a threshold; (ii) Noncoherent detection, i.e., computation of the magnitudes of the Fourier coefficients of the signal and then comparing a weighted sum of these coefficients to a threshold [1]; (iii) Modeling of the signal as a linear combination of known waveform with unknown coefficients, see e.g., [2]. In this paper we propose an adaptive detection scheme for transient signals, based on modeling of the signal as the impulse response of a rational transfer function. The coefficients of the transfer function are assumed unknown, and are estimated by the detector. It is shown in the paper that when the signal is absent, the parameter estimation problem becomes ill-posed. This difficulty is solved by introducing a constrained maximum likelihood estimator. A generalized likelihood ratio test (GLRT) is then introduced, and its statistical properties are analyzed. It is shown that the GLRT is approximately distributed as a quadratic form [3, ch. 29]. Under H_0, the quadratic form is central, while under H_1 it is noncentral and also contains a bias term. When the noise variance is unknown, the distribution of the GLRT involves ratios of quadratic forms (central under H_0 and non-central under H_1). Expressions for the weights, the noncentrality parameters and the bias are given as functions of the signal parameters.

The performance of the GLRT for the case of known noise variance is examined for some specific test cases, and theoretical performance curves are shown. Finally, the theoretical distributions of the likelihood ratio are compared to experimental distributions obtained by Monte-Carlo simulations.

2. THE GENERALIZED LIKELIHOOD RATIO DETECTOR

The detection problem considered in this section is as follows.

$$\begin{aligned} H_0 &: \ \underline{y} = \underline{v} \, , \\ H_1 &: \ \underline{y} = \underline{m} + \underline{v} \, , \end{aligned} \tag{1}$$

where $\underline{y}$, $\underline{m}$ and $\underline{v}$ are N-dimensional vectors, and where $\underline{v}$ is a zero mean white Gaussian noise whose variance σ^2 is assumed to be known. The signal $\underline{m}(\theta)$ is assumed to depend on a p-dimensional vector $\theta \varepsilon \Theta$, where $p < N$. We make the following assumptions on $\underline{m}(\theta)$:

(i) The functional dependence of $\underline{m}$ on θ is known, but the value of θ is unknown.

(ii) The $N \times p$ Jacobian matrix

$$M(\theta) \triangleq \frac{\partial \underline{m}(\theta)}{\partial \theta} \, , \tag{2}$$

exists for all $\theta \varepsilon \Theta$. The rank of this matrix will be denoted by $r(\theta)$. Note that $r(\theta)$ may vary for different values of θ, but we always have $r(\theta) < p$.

(iii) $\underline{m}(\theta) = 0$ if and only if $\theta = 0$. This assumption enables us to replace (3.1) by the equivalent detection problem:

$$\underline{y} = \underline{m}(\theta) + \underline{v}; \quad \begin{aligned} H_0 &: \ \theta = 0 \\ H_1 &: \ \theta \neq 0 \end{aligned} \tag{3}$$

(iv) $\underline{m}(\theta)$ belongs to the column space of $M(\theta)$ for all θ. Equivalently, the projection of $\underline{m}(\theta)$ on the column space of $M(\theta)$ is $\underline{m}(\theta)$ itself. Later we will show that this condition is satisfied when $\underline{m}(\theta)$ is the impulse response of a rational transfer function.

The generalized likelihood ratio is defined as

$$G = \frac{P_{\hat{\theta}}(\underline{y})}{P_0(\underline{y})} \tag{4}$$

where $P_\theta(\underline{y})$ is the joint probability density function of $\underline{y}$, given the parameter θ . The vector $\hat{\theta}$ is an estimate of θ whose exact nature will be specified later.

THE ESTIMATE $\hat{\theta}$

It would be natural to let $\hat{\theta}$ be the maximum likelihood estimate of θ . The log likelihood function is given by

$$\log P_\theta(\underline{y}) = -\frac{N}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}[\underline{y}-\underline{m}(\theta)]^T[\underline{y}-\underline{m}(\theta)]. \tag{5}$$

The maximum likelihood estimate is obtained by solving the likelihood equations

$$\frac{\partial \log P_\theta(\underline{y})}{\partial\theta} = \frac{1}{\sigma^2} M^T(\theta)[\underline{y}-\underline{m}(\theta)] = 0 . \tag{6}$$

Unfortunately, the matrix $M(\theta)$ need not have a full rank for all θ . In particular, $M(\theta)$ may be rank defficient at $\theta = 0$, i.e., under H_0 (later we will see that when the signal is a rational impulse response, the rank of M(0) is exactly half the number of parameters). Clearly, the number of independent equations in (6) is less than or equal to the rank of $M(\theta)$. Hence, when $M(\theta)$ is rank defficient, the number of independent equations in (6) will be smaller than the number of unknowns. Then the likelihoood equation will have an infinite number of solutions. Since equation (6) is nonlinear in θ , it is difficult to chaacterize the set of all possible solutions θ. However, it is very likely that the norms of the members of $\hat{\theta}$ will be unbounded, i.e.,

$$\sup_{\hat{\theta}\in\Theta} \|\hat{\theta}\| = \infty .$$

In this case any attempt to solve the likelihood equations (6) would lead to severe numerical problems, e.g., singularities and overflows. To circumvent this difficulty, we will therefore modify the log likelihood function (5) by

subtracting a term proportional to the norm of θ , i.e., we maximize

$$\log P_{\theta}(\underline{y}) - \frac{1}{2}\mu\theta^T\theta = -\frac{N}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}[\underline{y}-\underline{m}(\theta)]^T[\underline{y}-\underline{m}(\theta)] - \frac{1}{2}\mu\theta^T\theta \ . \tag{7}$$

This leads to the modified likelihood equations

$$\frac{1}{\sigma^2} M^T(\theta)[\underline{y}-\underline{m}(\theta)] - \mu\theta = 0 \ , \tag{8}$$

where μ is some positive scalar, serving to constrain the norm of $\hat{\theta}$.

FIRST ORDER APPROXIMATION OF $\hat{\theta}$

Let us assume that

$$\hat{\theta} = \theta + \Delta\hat{\theta} \ , \tag{9}$$

where $\Delta\hat{\theta}$ is sufficiently small, so that the following approximations can be made.

$$\underline{m}(\hat{\theta}) = \underline{m}(\theta) + M(\theta)\Delta\hat{\theta} \tag{10}$$

$$M(\hat{\theta}) = M(\theta) \ . \tag{11}$$

The likelihood equations (8) can then be approximated by

$$\frac{1}{\sigma^2} M^T(\theta)[\underline{m}(\theta)+\underline{v}-\underline{m}(\theta)-M(\theta)\Delta\hat{\theta}] - \mu\theta - \mu\Delta\hat{\theta} = 0. \tag{12}$$

Hence

$$\Delta\hat{\theta} = [M^T(\theta)M(\theta) + \sigma^2\mu I_p]^{-1}[M^T(\theta)\underline{v} - \sigma^2\mu\theta] \ . \tag{13}$$

Next we derive approximations for $M(\theta)\Delta\hat{\theta}$ and for $[\underline{y}-\underline{m}(\hat{\theta})]$. It is convenient to use the singular value decomposition (SVD) of $M(\theta)$, given by

$$M(\theta) = U(\theta) \begin{bmatrix} \Lambda(\theta) & 0 \\ 0 & 0 \end{bmatrix} V^T(\theta) \tag{3.14}$$

where $\Lambda(\theta)$ is a diagonal matrix of dimension $r(\theta) \times r(\theta)$, whose elements are all positive. The matrices $U(\theta)$ and $V(\theta)$ are orthogonal matrices of dimensions NxN and pxp respectively. We will usually omit the dependence on θ for notational convenience.

Substituting (14) in (13), we can express $\Delta\hat{\theta}$ as

$$\Delta\hat{\theta} = V \begin{bmatrix} (\Lambda^2+\sigma^2\mu I_r)^{-1}\Lambda & 0 \\ 0 & 0 \end{bmatrix} U^T\underline{v} - V \begin{bmatrix} \sigma^2\mu(\Lambda^2+\sigma^2\mu I_r)^{-1} & 0 \\ 0 & I_{p-r} \end{bmatrix} V^T\theta \quad . \tag{15}$$

Also,

$$M\Delta\hat{\theta} = U \begin{bmatrix} (\Lambda^2+\sigma^2\mu I_r)^{-1}\Lambda^2 & 0 \\ 0 & 0 \end{bmatrix} U^T\underline{v} - U \begin{bmatrix} \sigma^2\mu(\Lambda^2+\sigma^2\mu I_r)^{-1}\Lambda & 0 \\ 0 & 0 \end{bmatrix} V^T\theta \tag{16}$$

$$\underline{y}-\underline{m}(\hat{\theta}) = \underline{m}(\theta)+\underline{v}-\underline{m}(\theta)-M(\theta)\Delta\hat{\theta} = \underline{v} - M(\theta)\Delta\hat{\theta}$$

$$= U \begin{bmatrix} \sigma^2\mu(\Lambda^2+\sigma^2\mu I_r)^{-1} & 0 \\ 0 & I_{N-r} \end{bmatrix} U^T\underline{v} + U \begin{bmatrix} \sigma^2\mu(\Lambda^2+\sigma^2\mu I_r)^{-1}\Lambda & 0 \\ 0 & 0 \end{bmatrix} V^T\theta$$

$$= UAU^T\underline{v} + U\underline{b} \ , \tag{17}$$

where

$$A(\theta) \triangleq \begin{bmatrix} \sigma^2\mu(\Lambda^2 + \sigma^2\mu I_r)^{-1} & 0 \\ 0 & I_{N-r} \end{bmatrix} ; \ \underline{b}(\theta) \triangleq \begin{bmatrix} \sigma^2\mu(\Lambda^2+\sigma^2\mu I_r)^{-1}\Lambda & 0 \\ 0 & 0 \end{bmatrix} V^T\theta \ . \tag{18}$$

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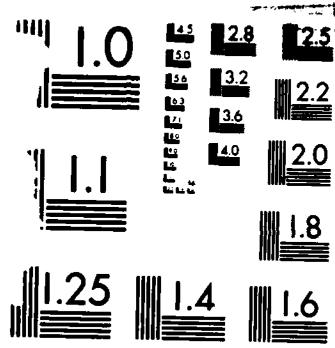

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FIRST ORDER APPROXIMATION OF THE LIKELIHOOD RATIO

The statistic G defined in (4) is given by

$$G = \exp\{\frac{1}{2\sigma^2}[\underline{y}^T\underline{y} - (\underline{y} - \underline{m}(\hat{\theta}))^T(\underline{y} - \underline{m}(\theta))]\} . \tag{19}$$

It is convenient to define the likelihood ratio as twice the logarithm of G, i.e., as

$$L_1 = \frac{1}{\sigma^2}\{\underline{y}^T\underline{y} - [\underline{y} - \underline{m}(\hat{\theta})]^T[\underline{y} - \underline{m}(\hat{\theta})]\} . \tag{20}$$

We now use the result (17) to derive first order approximations for L under H_0 and under H_1.

Under H_0:

$$L_1 \approx \frac{1}{\sigma^2}\{\underline{v}^T\underline{v} - \underline{v}^T U\ A^2\ U^T\underline{v}\} = \frac{1}{\sigma^2}\underline{w}^T[I_N-A^2]\underline{w} , \tag{21}$$

where

$$\underline{w} \overset{\Delta}{=} U^T\underline{v} . \tag{22}$$

Since U^T is orthogonal, $\underline{w}$ is a zero-mean white Gaussian noise with variance σ^2 . Also note that

$$I_N-A^2 = \begin{bmatrix} (\Lambda^2+\sigma^2\mu I_r)^{-2}(\Lambda^2+2\sigma^2\mu I_r)\Lambda^2 & 0 \\ 0 & 0 \end{bmatrix} . \tag{23}$$

Hence, under H_0 , L is approximately a central quadratic form

$$L_1 \approx \sum_{k=1}^{r} \frac{\lambda_k^2(\lambda_k^2 + 2\sigma^2\mu)}{(\lambda_k^2 + \sigma^2\mu)^2}(\frac{w_k}{\sigma})^2 . \tag{24}$$

Recall that the rank r and the singular values $\{\lambda_1, \lambda_2, \dots \lambda_r\}$ are those of the matrix $M(0)$. Note that under high signal-to-noise conditions, i.e., when $\sigma^2\mu << \lambda_k^2$, we have

$$L_1 \doteq \sum_{k=1}^{r} \left(\frac{w_k}{\sigma}\right)^2 \tag{25}$$

Thus, L_1 has approximately a central Chi-squared distribution with r degrees of freedom.

Under H_1:

$$\begin{aligned} L &= \frac{1}{\sigma^2} \{(\underline{v}+\underline{m})^T(\underline{v}+\underline{m}) - (AU^T\underline{v}+\underline{b})^T(AU^T\underline{v}+\underline{b})\} \\ &= \frac{1}{\sigma^2} \{(\underline{w}+\underline{q})^T(\underline{w}+\underline{q}) - (A\underline{w}+\underline{b})^T(A\underline{w}+\underline{b})\} \ , \end{aligned} \tag{26}$$

where

$$\underline{q} = U^T\underline{m} \ . \tag{27}$$

To further simplify this expression we note that by assumption (iv),

$$U \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} U^T \underline{m} = \underline{m} \ , \tag{28a}$$

or equivalently,

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \underline{q} = \underline{q} \ . \tag{28b}$$

The left hand side of (28a) is just the projection of $\underline{m}$ on the column space of M. Equation (28b) implies that only the first r components of q are nonzero. The same is true for $A\underline{b}$, as can be easily verified from (18). Therefore we can express L in the form

$$\begin{aligned} L &= \frac{1}{\sigma^2} \{\underline{w}^T(I_N-A^2)\underline{w} + 2(\underline{q}-A\underline{b})^T\underline{w} + \underline{q}^T\underline{q} - \underline{b}^T\underline{b}\} \\ &= \sum_{k=1}^{r} \beta_k\left[\frac{w_k}{\sigma} + \mu_k\right]^2 + \gamma \ , \end{aligned} \tag{29a}$$

where

$$\beta_k = \frac{\lambda_k^2(\lambda_k^2+2\sigma^2\mu)}{(\lambda_k^2+\sigma^2\mu)^2} \tag{29b}$$

$$\mu_k = \frac{\left[q_k - \frac{(\sigma^2\mu)^2\lambda_k}{(\lambda_k^2+\sigma^2\mu)^2}(V^T\theta)_k\right]}{\sigma\beta_k} \tag{29c}$$

$$\gamma = \frac{1}{\sigma^2}(\underline{q}^T\underline{q} - \underline{b}^T\underline{b}) - \sum_{k=1}^{r}\beta_k\mu_k^2 \tag{29d}$$

Hence, under H_1, L is a sum of a noncentral quadratic form and a bias term.

Note that for high SNR we have $\beta_k \doteq 1$, $\mu_k \doteq q_k/\sigma$ and $\gamma \doteq 0$. Therefore,

$$L_1 \doteq \sum_{k=1}^{r}[(\frac{w_k}{\sigma}) + (\frac{q_k}{\sigma})]^2 , \tag{30}$$

which has a non-central Chi-squared distribution with r degrees of freedom and non-centrality parameter

$$\underline{q}^T\underline{q}/\sigma^2 = \underline{m}^T\underline{m}/\sigma^2 = \text{SNR}. \tag{31}$$

3. THE GENERALIZED LIKELIHOOD RATIO DETECTOR: CASE OF UNKNOWN NOISE VARIANCE

In this section we consider a similar problem to the one discussed in the previous section, except that we now assume the noise variance σ^2 to be unknown. Assumptions (i)-(iv) will be made, as before.

The generalized likelihood ratio is now defined as

$$G = \frac{P_{\hat{\theta},\hat{\sigma}^2}(\underline{y})}{P_{0,\hat{\sigma}_0^2}(\underline{y})} \tag{32}$$

where $P_{\theta,\sigma^2}(\underline{y})$ is the joint probability density of function of $\underline{y}$, given the parameters θ and σ^2. The quantities $\hat{\theta}$ and $\hat{\sigma}^2$ are estimates of θ and σ^2 respectively, while $\hat{\sigma}_0^2$ is an estimate of σ^2, given that $\theta = 0$. These estimates are discussed below.

THE ESTIMATE $\hat{\sigma}_0^2$

Under the assumption that $\theta = 0$, the log likelihood function is given by

$$\log P_{0,\sigma^2}(\underline{y}) = -\frac{N}{2}\log 2\pi - \frac{N}{2}\log\sigma^2 - \frac{1}{2\sigma^2}\underline{y}^T\underline{y} \tag{33}$$

To find $\hat{\sigma}_0^2$ we differentiate (33), equate the derivative to zero and solve for σ^2. This yields

$$\hat{\sigma}_0^2 = \frac{1}{N}\underline{y}^T\underline{y} \tag{34}$$

THE ESTIMATES $\hat{\sigma}^2$, $\hat{\theta}$

As explained in the previous section, the likelihood function has to be modified by a constraint term, in order to avoid unboundedness of $\hat{\theta}$ when H_0 is true. We therefore use,

$$\log P_{\theta,\sigma^2}(\underline{y}) - \frac{1}{2}\mu\theta^T\theta$$

$$= -\frac{N}{2}\log 2\pi - \frac{N}{2}\log\sigma^2 - \frac{1}{2\sigma^2}[\underline{y} - \underline{m}(\theta)]^T[\underline{y} - \underline{m}(\theta)] - \frac{1}{2}\mu\theta^T\theta \tag{35}$$

The resulting likelihood equations are

$$\hat{\sigma}^2 = \frac{1}{N}[\underline{y} - \underline{m}(\hat{\theta})]^T[\underline{y} - \underline{m}(\hat{\theta})] \tag{36a}$$

$$M^T(\hat{\theta})[\underline{y} - \underline{m}(\hat{\theta})] - \mu\hat{\sigma}^2\hat{\theta} = 0 \tag{36b}$$

When first order approximates are assumed, as in the previous section, it is found that

$$\Delta\hat{\theta} = [M^T(\theta)M(\theta) + \sigma^2\mu I_p]^{-1}[M^T(\theta)\underline{v} - \sigma^2\mu\theta]. \tag{37}$$

This is the same expression as (13), i.e., a small error in $\hat{\sigma}^2$ does not change the estimate $\hat{\theta}$ up to a first order.

FIRST ORDER APPROXIMATION OF THE LIKELIHOOD RATIO

The statistic G defined in (32) is given by

$$G = \frac{(2\pi\hat{\sigma}^2)^{-N/2}\exp\{-\frac{N}{2}\}}{(2\pi\hat{\sigma}_0^2)^{-N/2}\exp\{-\frac{N}{2}\}} = [\frac{\hat{\sigma}^2}{\hat{\sigma}^2}]^{N/2}. \tag{38}$$

It is convenient to define the likelihood ratio as the following monotone function of G:

$$L_2 = G^{2/N} - 1 = \frac{\hat{\sigma}_0^2}{\hat{\sigma}^2} - 1 = \frac{\hat{\sigma}_0^2 - \hat{\sigma}^2}{\hat{\sigma}^2} = \frac{\hat{\sigma}_0^2 - \hat{\sigma}^2}{\hat{\sigma}_0^2 - (\hat{\sigma}_0^2 - \hat{\sigma}^2)} \tag{39}$$

where

$$\hat{\sigma}_0^2 - \hat{\sigma}^2 = \{\underline{y}^T\underline{y} - [\underline{y} - \underline{m}(\hat{\theta})]^T[\underline{y} - \underline{m}(\hat{\theta})]\}/\sigma^2$$

$$\hat{\sigma}_0^2 = \underline{y}^T\underline{y}/\sigma^2 \tag{40}$$

Using the derivations in the previous section we can write

Under H_0:

$$L_2 \doteq \frac{\{\underline{w}^T[I_N - A^2]\underline{w}\}/\sigma^2}{\underline{w}^T\underline{w}/\sigma^2 - \{\underline{w}^T[I_N - A^2]\underline{w}\}/\sigma^2} =$$

$$= \frac{\sum_{k=1}^{r} \frac{\lambda_k(\lambda_k + 2\sigma^2\mu)}{(\lambda_k^2 + \sigma^2\mu)^2}\left(\frac{w_k}{\sigma}\right)^2}{\sum_{k=1}^{N}\left(\frac{w_k}{\sigma}\right)^2 - \sum_{k=1}^{r}\frac{\lambda_k^2(\lambda_k^2 + 2\sigma^2\mu)}{(\lambda_k^2 + \sigma^2\mu)^2}\left(\frac{w_k}{\sigma}\right)^2} \tag{41}$$

where $\underline{w}$, A, λ_k are as defined in section 2. Note that under high signal-to-noise conditions, i.e., when $\sigma^2\mu \ll \lambda_k^2$, we have

$$L_2 \doteq \frac{\sum_{k=1}^{r}\left(\frac{w_k}{\sigma}\right)^2}{\sum_{k=r+1}^{N}\left(\frac{w_k}{\sigma}\right)^2} \tag{42}$$

Thus, at high SNR L_2 is proportional to a random variable with a central $\mathscr{F}_{r,N-r}$ distribution.

Under H_1:

$$L_2 \doteq \frac{\{(\underline{w}+\underline{q})^T(\underline{w}+\underline{q}) - (A\underline{w}+\underline{b})^T(A\underline{w}+\underline{b})\}/\sigma^2}{(\underline{w}+\underline{q})^T(\underline{w}+\underline{q})/\sigma^2 - \{(\underline{w}+\underline{q})^T(\underline{w}+\underline{q}) - (A\underline{w}+\underline{b})^T(A\underline{w}+b)\}/\sigma^2}$$

$$= \frac{\{\sum_{k=1}^{r}\beta_k\left(\frac{w_k}{\sigma} + \mu_k\right)^2 + \gamma\}}{\sum_{k=1}^{N}\left(\frac{w_k + q_k}{\sigma}\right)^2 - \{\sum_{k=1}^{r}\beta_k\left(\frac{w_k}{\sigma} + \mu_k\right)^2 + \gamma\}} \tag{43}$$

where $\underline{w}$, $\underline{q}$, A, $\underline{b}$, β_k, μ_k and γ are as defined in section 2.

Under high SNR conditions we have: $\beta_k \doteq 1$, $\mu_k \doteq q_k/\sigma$, $\gamma \doteq 0$. Thus,

$$L_2 \doteq \frac{\sum_{k=1}^{r}[(\frac{w_k}{\sigma}) + (\frac{q_k}{\sigma})]^2}{\sum_{k=r+1}^{N}[(\frac{w_k}{\sigma})+(\frac{q_k}{\sigma})]^2} \tag{44}$$

Thus, at high SNR L_2 is proportional to a random variable with a noncentral $\mathcal{F}_{r,N-r}$ distribution with noncentrality parameter

$$\frac{\underline{q}^T\underline{q}}{\sigma^2} = \frac{\underline{m}^T\underline{m}}{\sigma^2} = \text{SNR} \tag{45}$$

Note also that for high SNR and $N \to \infty$ we have

H_0 : $L_2 \sim \chi_r^2(\text{SNR})$, central Chi-squared with r degrees of freedom

(46)

H_1: $L_2 \sim \chi_r^2(\text{SNR})$, non-central Chi-squared and non-centrality parameter SNR

In general (when the SNR is not sufficiently large) L_2 will be a monotonic function of a ratio of quadratic forms

$$L_2 = \frac{\bar{L}_2}{1 + \bar{L}_2} \tag{47}$$

where

$$H_0 : \bar{L}_2 = \frac{\sum_{k=1}^{r} \frac{\lambda_k^2(\lambda_k^2+2\sigma^2\mu)}{(\lambda_k^2 + \sigma^2\mu)^2}(\frac{w_k}{\sigma})^2}{\sum_{k=1}^{N}(\frac{w_k}{\sigma})^2}, \quad \text{central ratio of quadratic forms}$$

$$\tag{48}$$

$$H_1 : \bar{L}_2 = \frac{\sum_{k=1}^{r} \beta_k(\frac{w_k}{\sigma} + \mu_k)^2 + \gamma}{\sum_{k=1}^{N} (\frac{w_k+q_k}{\sigma})^2}, \quad \text{noncentral ratio of quadratic forms.}$$

4 THE GLRT IN THE CASE OF RATIONAL SIGNAL

Next we consider the case where the signal $\underline{m}(\theta)$ can be modeled as the impulse response of a rational transfer function, i.e.

$$m_k = \frac{1}{2\pi j}\oint \frac{b(z)}{a(z)} z^{k-1} dz \ , \quad 1 < k < N \ , \tag{49}$$

where

$$\frac{b(z)}{a(z)} = \frac{b_1 z^{n-1} + \dots + b_n}{z^n + a_1 z^{n-1} + \dots + a_n} \ . \tag{50}$$

The polynomials a(z) and b(z) are assumed to be coprime; a(z) is assumed to be stable. The parameter vector is the 2n-dimensional vector

$$\theta = [a_1, \ a_2, \dots, a_n, b_1, b_2, \dots, b_n]^T . \tag{51}$$

In order to comply with assumption (iii), we assume that
$b(z) = 0 \rightarrow a(z) = z^n$, i.e., if all $\{b_k\}$ $\{b_k\}$ are zero, then all $\{a_k\}$ are necessarily zero.

First we show that assumption (iv) is satisfied for the model (49). We have

$$\frac{\partial m_k}{\partial a_\ell} = -\frac{1}{2\pi j}\oint \frac{z^{n-\ell} b(z)}{a^2(z)} z^{k-1} dz \tag{52a}$$

$$\frac{\partial m_k}{\partial b_\ell} = \frac{1}{2\pi j}\oint \frac{z^{n-\ell}}{a(z)} z^{k-1} dz \ . \tag{52b}$$

Hence

$$\sum_{\ell=1}^{n} b_\ell \frac{\partial m_k}{\partial b_\ell} = \frac{1}{2\pi j}\oint \frac{\sum_{\ell=1}^{n} b_\ell z^{n-\ell}}{a(z)} z^{k-1} dz = m_k \ , \tag{53}$$

so that $\underline{m}(\theta)$ is a linear combination of the columns of $M(\theta)$, as required.

Next let us compute the rank of $M(\theta)$ for $\theta=0$ and for $\theta \neq 0$. For $\theta = 0$ we have:

$$\frac{\partial m_k}{\partial a_\ell} = 0; \quad \frac{\partial m_k}{\partial b_\ell} = \frac{1}{2\pi j}\oint z^{k-\ell-1}dz = \delta_{k,\ell} \quad . \tag{54}$$

Hence

$$M(0) = \begin{bmatrix} 0 & I_n \\ 0 & 0 \end{bmatrix} \begin{matrix} \} \; n \\ \} \; N-n \end{matrix} \longrightarrow r(0) = n \; . \tag{55}$$

Next we show that $r(\theta) = 2n$ for all $\theta \neq 0$. Assume the converse, i.e., that for some nonzero vector $\zeta = [c_1, c_2, \ldots, c_n, d_1, d_2, \ldots, d_n]^T$ we have

$$0 = M(\theta)\,\xi = \{-\frac{1}{2\pi j}\oint \frac{(\sum_{\ell=1}^{n} c_\ell z^{n-\ell})b(z)}{a^2(z)} z^{k-1}dz$$

$$+ \frac{1}{2\pi j}\oint \frac{(\sum_{\ell=1}^{n} d_\ell z^{n-\ell})}{a(z)} z^{k-1}dz\}_{1<k<N}$$

$$= \{\frac{1}{2\pi j}\oint \frac{d(z)a(z)-c(z)b(z)}{a^2(z)} z^{k-1}dz\}_{1<k<N} \; . \tag{56}$$

This means that

$$d(z)a(z) - c(z)b(z) = 0 \tag{57}$$

for some polynomials $\{c(z),d(z)\}$, contradicting the assumed coprimeness of $\{a(z),b(z)\}$

Finally, we note that in the case of a rational signal, the likelihood ratio under H_0 is given by (cf. (24) and (55))

$$L = \frac{1+2\sigma^2\mu}{(1+\sigma^2\mu)^2} \sum_{k=1}^{n} (\frac{w_k}{\sigma})^2 \; . \tag{58}$$

Therefore, under H_0 , the likelihood ratio is proportional to a random variable whose distribution is χ^2 with n degress of freedom.

Similarly, under L_2 is given by

$$L_2 \doteq \frac{\dfrac{1+2\sigma^2\mu}{(1+\sigma^2\mu)^2}\sum_{k=1}^{n}\left(\frac{w_k}{\sigma}\right)^2}{\sum_{k=n+1}^{N}\left(\frac{w_k}{\sigma}\right)^2+\left(1-\dfrac{1+2\sigma^2\mu}{(1+\sigma^2\mu)^2}\right)\sum_{k=1}^{n}\left(\frac{w_k}{\sigma}\right)^2}$$

$$= \frac{1+2\sigma^2\mu}{(1+\sigma^2\mu)^2}\;\frac{F}{1+\left(1-\dfrac{1+2\sigma^2\mu}{(1+\sigma^2\mu)^2}\right)F} \tag{59}$$

where

$$F = \frac{\sum_{k=1}^{n}\left(\frac{w_k}{\sigma}\right)^2}{\sum_{k=n+1}^{N}\left(\frac{w_k}{\sigma}\right)^2} \tag{60}$$

Thus, under H_0 L_2 is afunction of a $\mathscr{F}_{n,N-n}$ distribution. Note that when $\sigma^2\mu \ll 1$

$$L_2 \doteq F \sim \mathscr{F}_{n,N-n} \tag{61}$$

Equations (58), (59) can be used to compute the thresholds of the detector for a specified probability of false alarm P_{FA}.

Under H_1: the GLRT L_1 is a quadratic form (cf. (28)-(31)) and L_2 is a function of a ratio of quadratic forms (cf. (47), (48)). The probability distribution function of the first can be computed, given the weights $\{\beta_k\}$ and the noncentralities $\{\mu_k\}$, using one of the methods suggested in [3]. The probability distribution function of the second is difficult to compute in general. In the case of high SNR L_2 has approximately an $\mathscr{F}_{n,\, N-n}$ distribution, cf. (42), (44):

$$\begin{aligned} H_0 &: L_2 \sim \mathscr{F}_{n,N-n} \quad \text{, central F-distribution} \\ H_1 &: L_2 \sim \mathscr{F}_{n,\,N-n}(SNR) \quad \text{, non-central F-distribution} \end{aligned} \tag{62}$$

5. NUMERICAL EXAMPLES

In this section we illustrate the performance of GLRT in two test cases. In both cases the signal is modeled by the second-order transfer function

$$\frac{b(z)}{a(z)} = \frac{b_1 z}{z^2+a_1 z+a_2} \ . \tag{63}$$

In the first test case $a(z) = z^2-1.4z + 0.95$, while in the second case $a(z) = z^2-1.3z + 0.75$. The parameter b_1 was computed according to the desired signal energy in each example. The parameter b_2 is identically zero.

As is clear from the previous section, the GLRT for the case of unknown variance approaches the one for the case of known noise variance as the number of data N becomes sufficiently large. Therefore, and due to the difficulty in computing the distribution of a ratio of quadratic forms, we settled for testing the case of known variance only.

Besides the GLRT, we examined two other detectors for comparison. The first is the matched filter (MF)

$$L_{MF} \triangleq \frac{1}{\sigma^2} \underline{m}^T(\theta)\underline{y} \ , \tag{64}$$

where $\underline{m}(\theta)$ is the true signal. The matched filter corresponds to a situation when the signal waveform is known, and represents an upper bound on the performance of any detector for the detection problem (1). The distribution of L_{MF} under H_0 is normal with zero mean and variance $\underline{m}^T\underline{m}/\sigma^2$, while the distribution under H_1 is normal with both mean and variance equal to $\underline{m}^T\underline{m}/\sigma^2$.

The second detector is the energy detector (ED)

$$L_{ED} = \frac{1}{\sigma^2} \underline{y}^T\underline{y} \ . \tag{65}$$

The energy detector does not make any assumption on the signal, hence it represents a lower bound on the performance of any reasonable detector. It is not difficult to show that the distribution of L_{ED} is χ^2 with N degrees of

freedom. Under H_0 the distribution is central, while under H_1 it is noncentral, with noncentrality parameter $\underline{m}^T\underline{m}/\sigma^2$.

First we compute the theoretical detection probabilities of the three detectors as a function of the signal to noise ratio, keeping the probability of false alarm P_{FA} fixed. For each detector, the corresponding the threshold was computed by

$$t = F_0^{-1}(1-P_{FA}) , \tag{66}$$

where $F_0(\ )$ denotes the cumulative distribution fuction of the likelihood ratio in question under H_0 (normal for MF, χ_n^2 for GLRT and χ_N^2 for ED). The detection probability was then computed by

$$P_D = 1 - F_1(t) , \tag{67}$$

where $F_1(\ )$ is the cumulative distribution function of the corresponding likelihood ratio under H_1 . The quadratic form distribution of the GLRT was computed by numerical evaluation of the inverse Fourier transform of the characteristic function.

Figure 1 shows the theoretical detection probabilities of the three detectors for the narrowband case as a function of the SNR, with $P_{FA} = 10^{-2}$ and $N = 60$. Figure 2 depicts a similar case, except that $P_{FA} = 10^{-3}$. Figures 3 and 4 show the corresponding results for the medium band case, with $N = 10$. The SNR is defined here as the ratio of the <u>total</u> signal energy to the noise variance, i.e., $\underline{m}^T\underline{m}/\sigma^2$. Note that the actual SNR is $\underline{m}^T\underline{m}/(N\sigma^2)$. As expected, the performance of the GLRT is between those of the matched filter and the energy detector. In the narrowband case, the GLRT is much better than the ED, while in the medium band case the GLRT is approximately in the middle between ED and MF.

Next, we tested the behavior of the GLRT by Monte Carlo simulations, and compared the experimental distributions to the theoretical ones. In each case, 1000 Monte-Carlo simulations were run. The constraint parameter μ was set to 5 in all cases. Figure 5 shows the theoretical and experimental

distributions of the GLRT under H_0. The number of data points N was 50. As we see, the experimental distribution matches the theoretical one quite well.

Figure 6 shows the theoretical and experimental distributions under H_1 for the narrowband signal. The number of data points was again 50 and the SNR was 13 dB. Again, the two distributions are fairly close, except for the "bump" in the experimental distribution at low values of L. Observing the individual Monte-Carlo runs, we found that in some of them the constraint term "pulled" the estimate $\hat{\theta}$ to relatively low values. This phenomenon is not accounted for by first-order approximations derived in section 2, and serves to explain the difference between the two curves.

Figure 7 shows the two distributions under H_1 for the medium band case. The number of data points was 20 and the SNR was 16 dB. As we see, the "bump" near the origin is now larger than in the previous case. This means that the effect of the constraint term is now more severe, "pulling" $\hat{\theta}$ to low values more often. The bump causes an approximately constant difference between the two curves for $L > 30$.

5. CONCLUSIONS

We presented an adaptive scheme for detecting transient waveforms of unknown characteristics in white Gaussian noise. The detector is based on a generalized likelihood ratio test, and uses a constrained maximum likelihood estimation of the signal parameters. Approximate expressions were derived for the distributions of the likelihood ratio under H_0 and under H_1. It was shown that in the case of known noise variance the likelihod ratio is distributed as a quadratic form with the number of degrees of freedom equal to (or less than) the number of unknown parameters. By comparison, the energy detector is distributed as a χ^2 with the number of degrees of freedom equal to the number of data points. Thus, the GLRT performs considerably better than the energy detector in cases where the number of unknown parameters is much smaller than the number of data points. In the case of unknown noise variance the GLRT involves ratios of quadratic forms.

When the theoretical distributions were compared to experimental ones, some discrepancy was observed. This discrepancy is attributed in part to the effect of the constraint term in the maximum likelihood estimator. For narrowband signals the discrepancy is small, while for medium or broadband signals it may be quite large.

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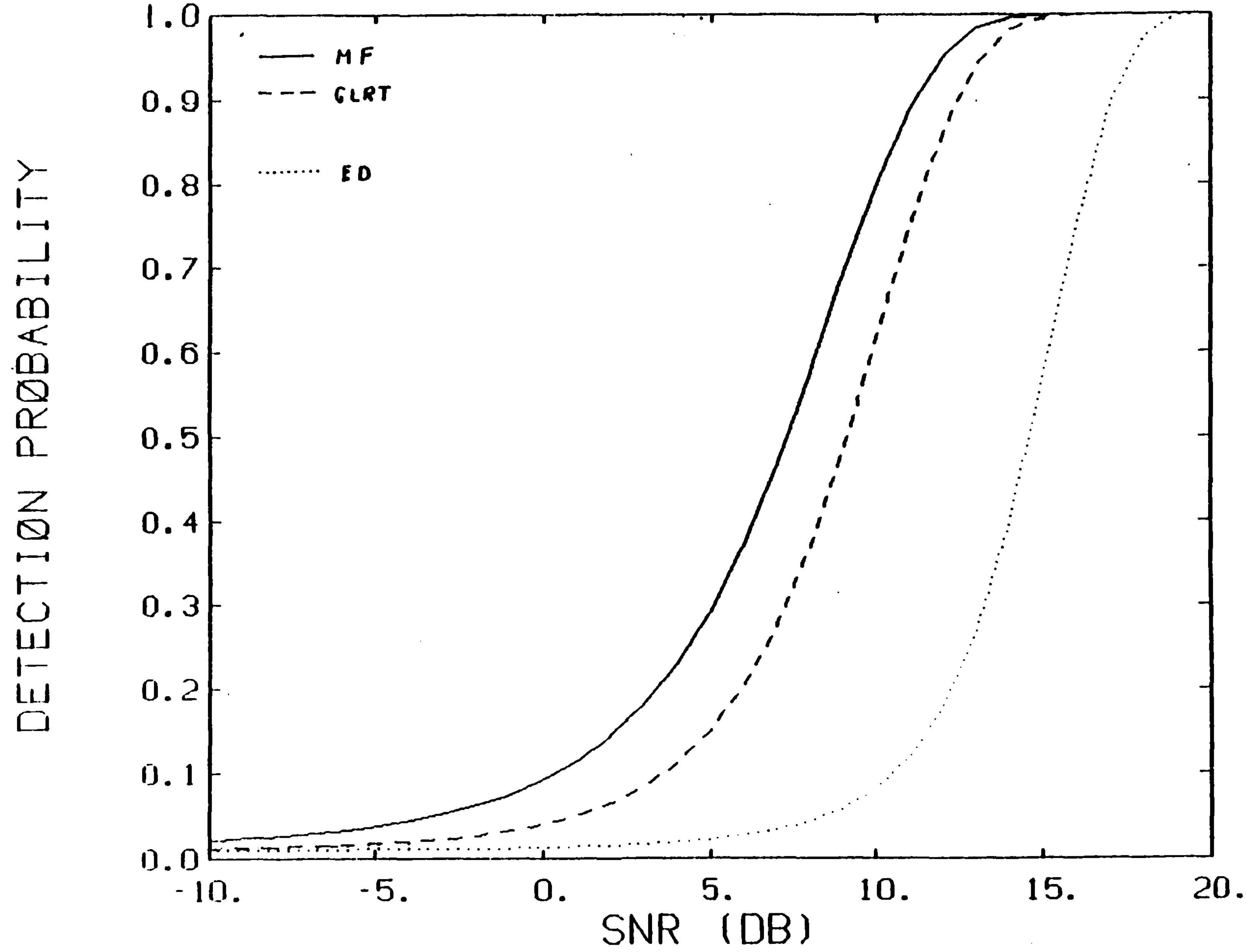

Figure 1. Performance Curves for Narrow Band Signal, $P_{FA} = 10^{-2}$, N = 60

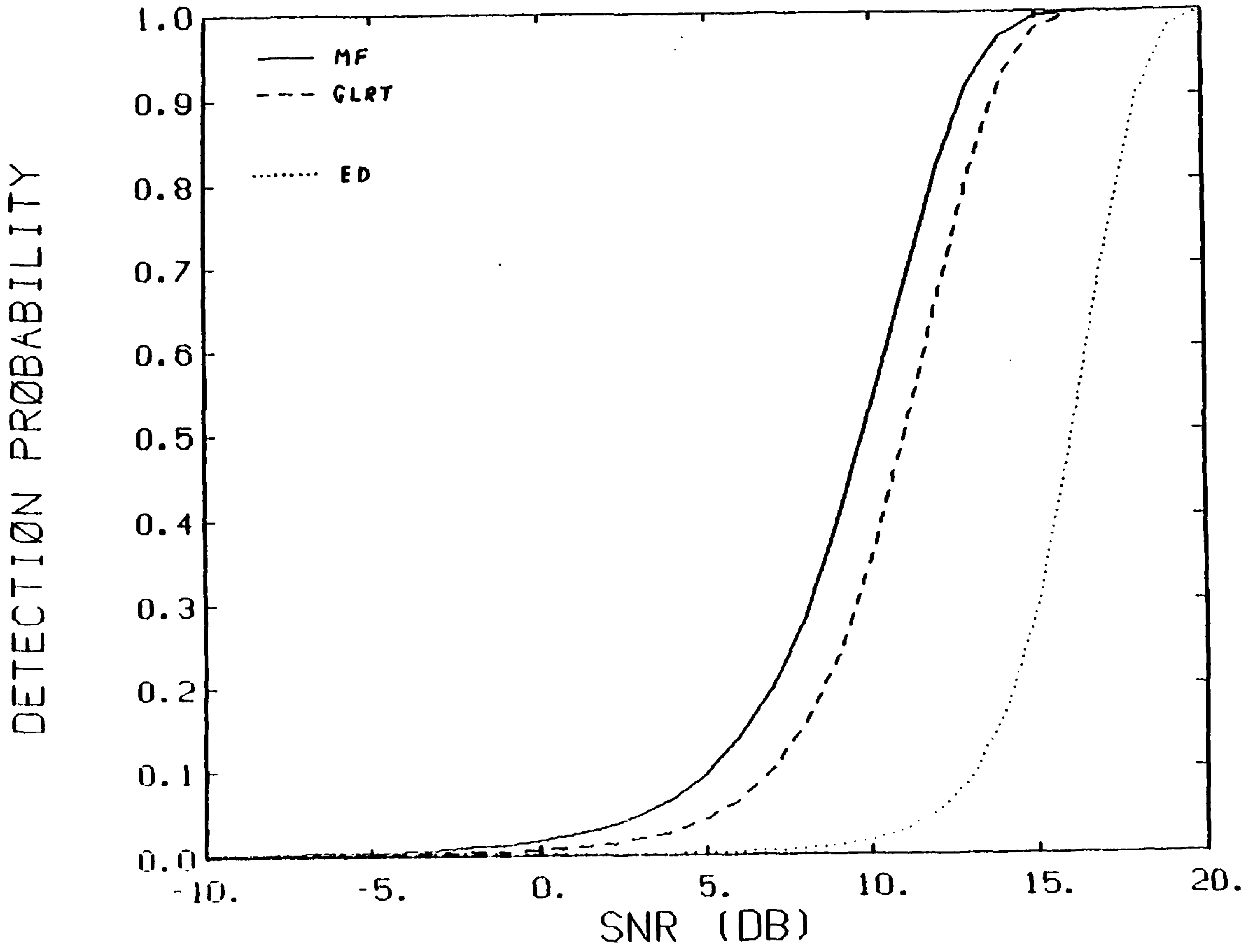

Figure 2. Performance Curves for Narrowband Signal, $P_{FA} = 10^{-3}$, N = 60

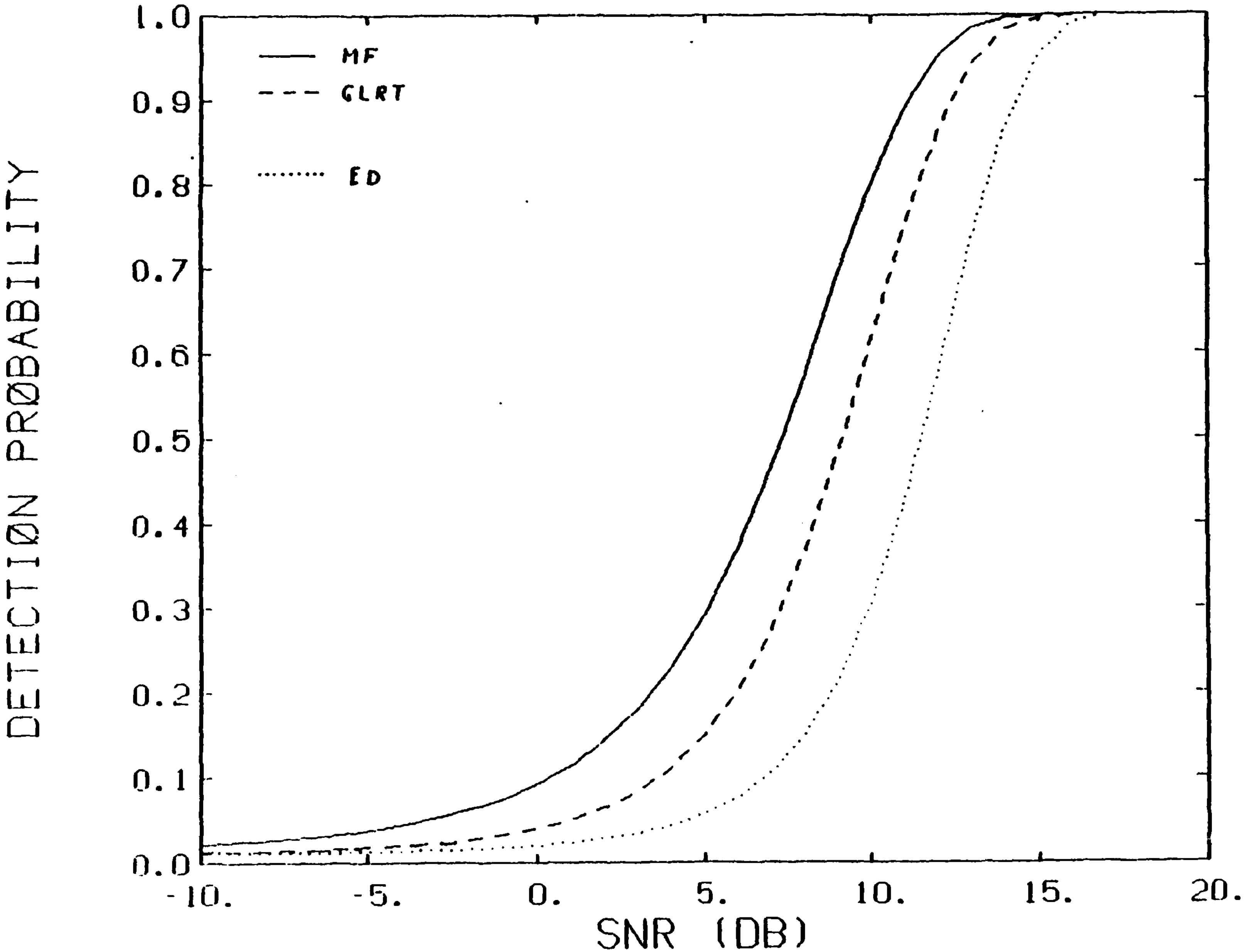

Figure 3. Performance Curves for Medium Band Signal, $P_{FA} = 10^{-2}$, N = 10

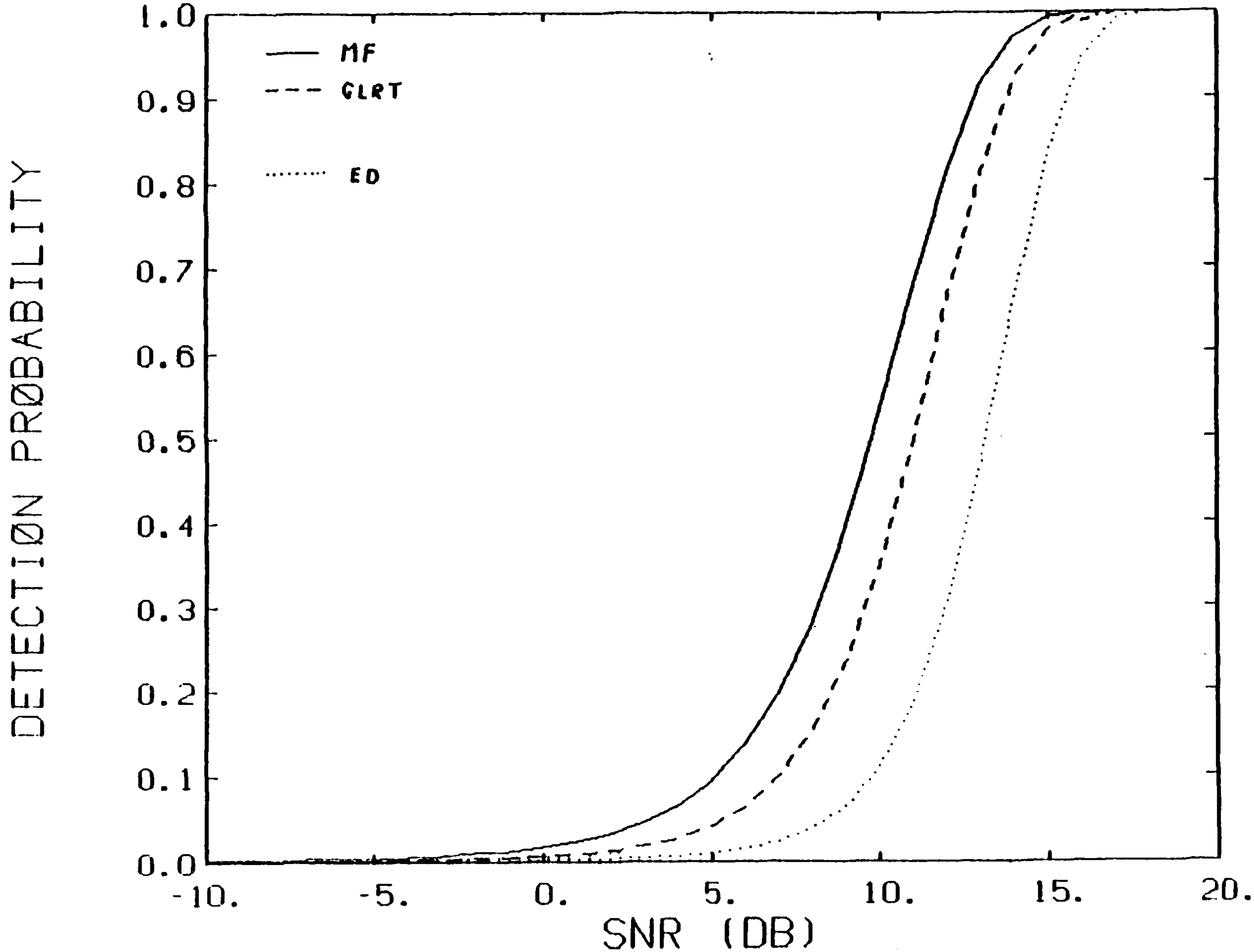

Figure 4. Performance Curves for Medium Band Signal, $P_{FA} = 10^{-3}$, N = 10

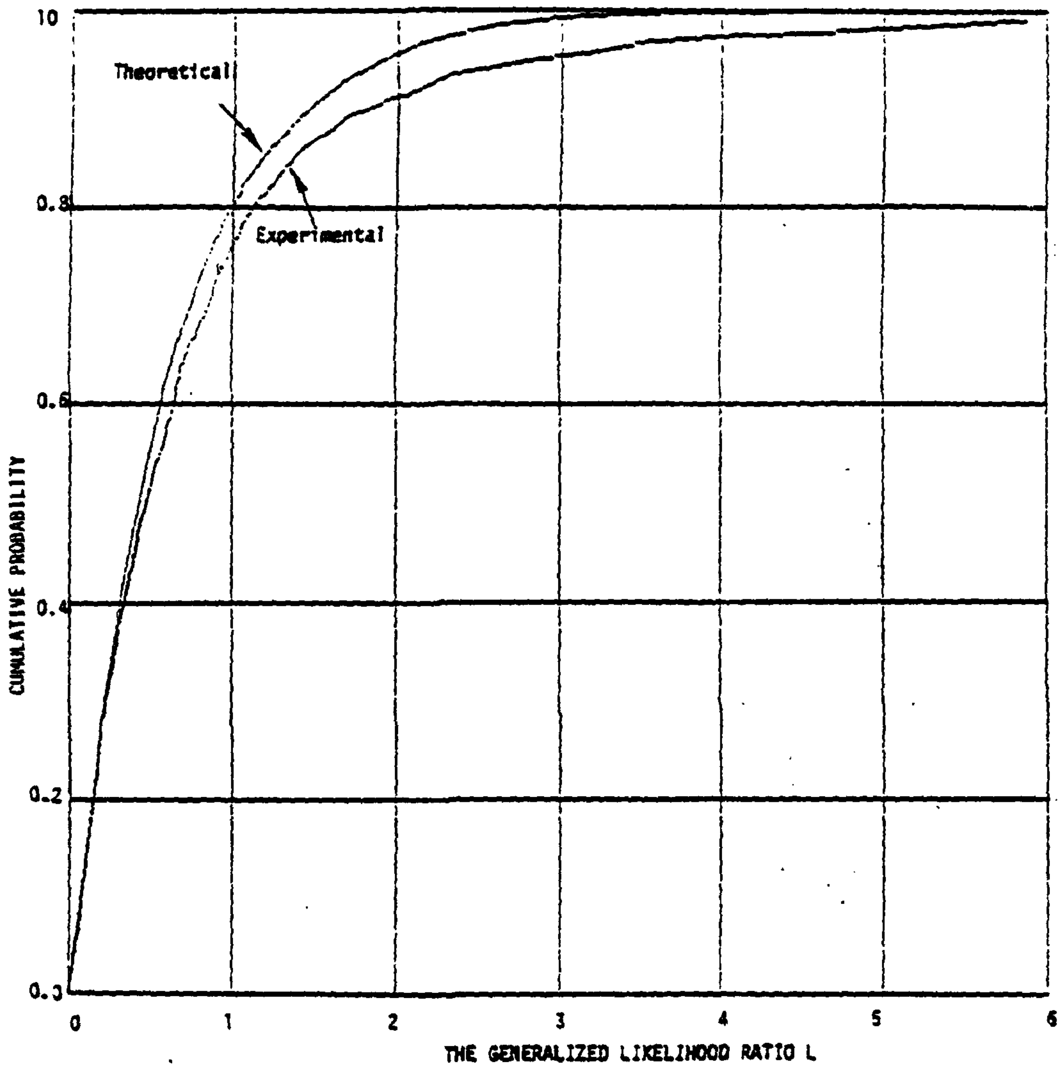

Figure 5. Cumulative Distribution of the GLRT Under H_0

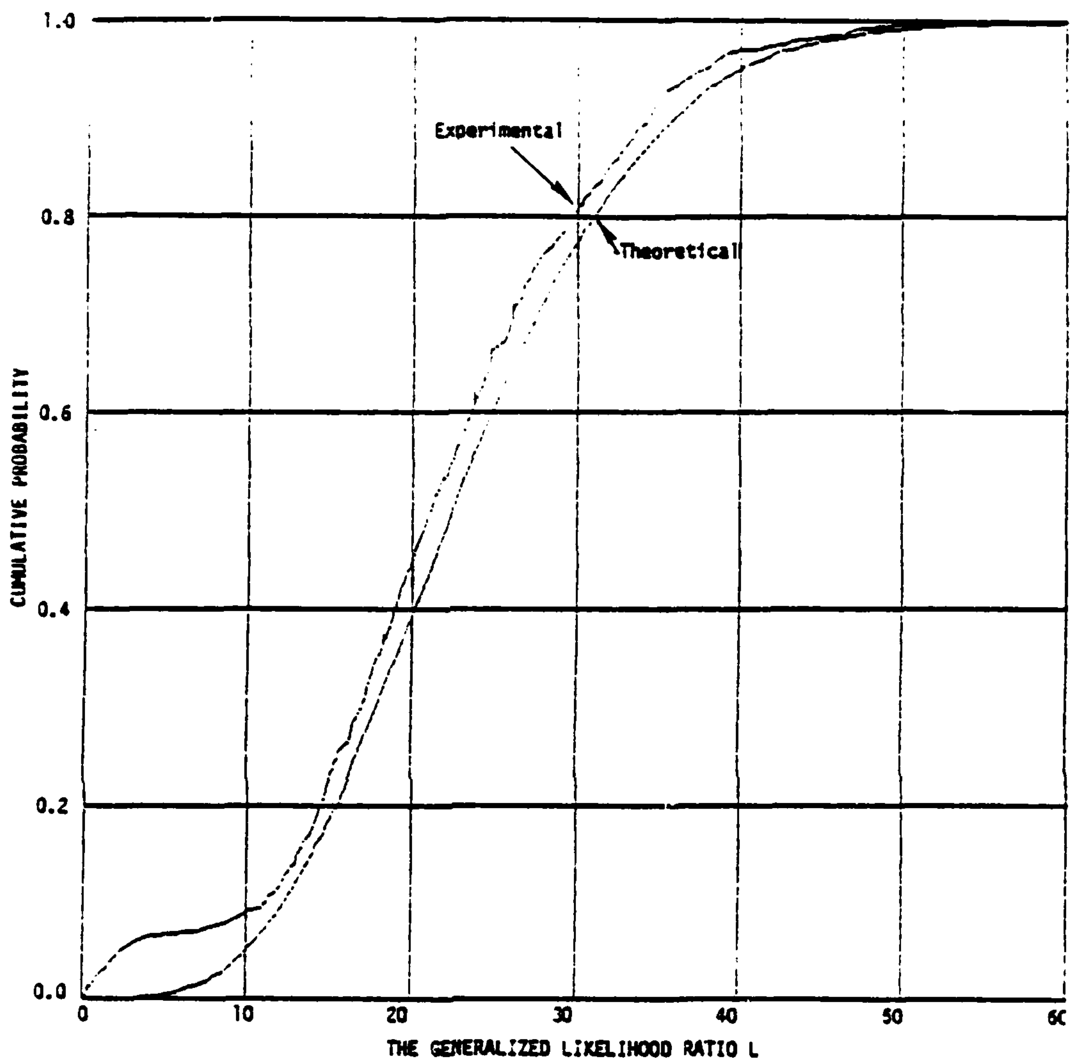

Figure 6. Cumulative Distribution of the GLRT Under H_1 - Narrowband Signal

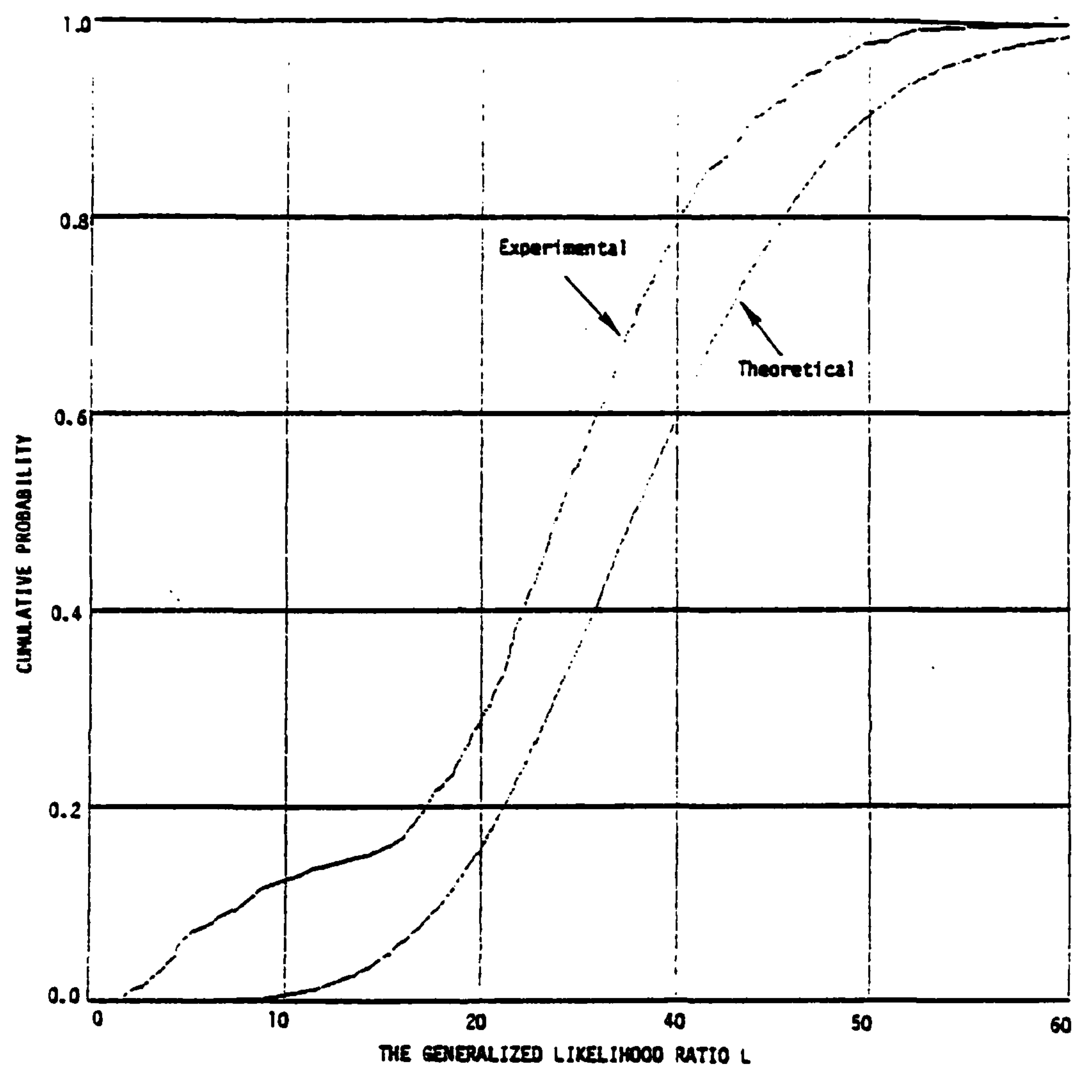

Figure 7. Cumulative Distribution of the GLRT Under H_1 - Medium Band Signal

APPENDIX J

BOUNDS ON THE ACCURACY OF ARMA PARAMETER ESTIMATION METHODS BASED ON SAMPLE COVARIANCES

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BOUNDS ON THE ACCURACY OF ARMA PARAMETER ESTIMATION METHODS BASED ON SAMPLE COVARIANCES

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ABSTRACT

The paper considers the asymptotic accuracy of ARMA parameter estimation methods based on a fixed number of sample covariances. A general expression for the error covariance of the ARMA parameter estimates is presented. It is shown that the error covariance is always greater than a certain lower bound, and that this lower bound is strictly greater than the Cramer-Rao bound. An explicit ARMA estimation technique that asymptotically achieves the bound is presented. Finally, it is shown that this lower bound approaches the Cramer-Rao bound as the number of sample covariances tends to infinity.

This work was supported by the Army Research Office under contract No. DAAG29-83-C-0027.

I. INTRODUCTION

The problem of estimating the parameters of ARMA processes has been treated extensively in the statistical and engineering literature. ARMA parameter estimation techniques can be classified into two general categories: methods that use the data directly, and methods that apply some preliminary transformations to the data. Among the methods in the first class we mention in particular the exact maximum likelihood method, and its many approximations [1]-[4]. Typically, such approximations are aimed at preserving the asymptotic properties of the maximum likelihood method, namely consistency, asymptotic efficiency and asymptotic normality, while reducing its computational complexity. Among the second class, probably the most common approach is to transform the data into a finite set of sample covariances and then estimate the ARMA parameters from these sample covariances. References [5]-[7] include examples of this class of estimation techniques. Most of the system identification techniques used in practice are based, either explicitly or implicitly, on sample covariances.

In the special case of autoregressive (AR) processes, methods of the second class are known to be asymptotically equivalent to the maximum likelihood method [8]. The first $p+1$ sample covariances (where p is the order of the AR process in question), while not being a sufficient statistic for the AR parameters [9], are known to yield asymptotically efficient estimates of the parameters via the Yule-Walker equations [8]. ARMA parameter estimation methods based on sample covariances are known to be less efficient than maximum likelihood ARMA estimates. This phenomenon can be explained as follows. At least $p+q+1$ sample covariances are needed to estimate the $p+q+1$ components of θ. However, it was recently shown that only the sample covariances of orders $0 \leq \ell \leq p-q$ are asymptotically efficient estimates of the corresponding true covariances, while sample covariances of higher orders are not asymptotically efficient [10]. Since $p+q+1 > p-q+1$ for all $q > 0$, some loss of efficiency of the ARMA estimates based on the sample covariances is inevitable.

The discussion above naturally raises the question: what is the asymptotic accuracy of ARMA parameter estimation techniques based on sample

covariances? Partial answers to this question can be found in the literature. The accuracy of a particular estimation technique, the so-called high order Yule-Walker method, was considered in [13],[14]. Note that these references treat only the accuracy of the estimates of the AR part of the ARMA parameters. The best accuracy achievable by any estimator based on sample covariances was studied by Bruzzone and Kaveh [15]-[18]. They defined a scalar measure of accuracy and computed its value for various examples. These results verified the inefficiency of ARMA estimates based on a finite number of sample covariances.

In this paper we present a fairly complete set of results on the asymptotic accuracy of ARMA parameter estimation techniques based on sample covariances. In section 3 we derive general asymptotic expressions for the asymptotic bias and covariance of a general class of ARMA parameter estimates based on sample covariances. In section 4 we briefly review previous results on the accuracy of ARMA parameter estimates based on a finite number of sample covariances, and present new proofs of these results. A lower bound for the error covariance matrix is presented and is shown to be strictly larger than the Cramer-Rao bound. A specific estimator which achieves this bound is also presented. In section 5 we prove that this lower bound approaches the Cramer-Rao bound as the number of sample covariances tends to infinity. This result is commonly assumed in the literature, but has not been formally proven. Finally, we illustrate the theoretical results by some numerical examples.

In the next section we define the problem under consideration and introduce some necessary notation.

2. PROBLEM STATEMENT

A Gaussian autoregressive moving-average (ARMA) process is defined by the difference equation

$$y_t = -\sum_{k=1}^{p} a_k y_{t-k} + u_t + \sum_{k=1}^{q} b_k u_{t-k} \quad , \tag{1}$$

where $\{u_t\}$ is a zero mean Gaussian white noise with variance σ_u^2 . The polynomials

$$a(z) = 1 + a_1 z + \ldots + a_p z^p; \qquad b(z) = 1+b_1 z + \ldots + b_q z^q ,$$

are required to satisfy the following conditions:

(i) $a(z) \neq 0$, $b(z) \neq 0$ for all $|z| \leq 1$, i.e., all the roots of these polynomials are outside the unit circle;

(ii) $a_p \neq 0$, $b_q \neq 0$;

(iii) $a(z)$ and $b(z)$ are relatively prime, i.e., they have no common roots.

Conditions (ii) and (iii) imply minimality of the description (1) of the given process. Under these conditions, the (p+q+1)-dimensional parameter vector

$$\theta = [\sigma_u, a_1, \ldots, a_p, b_1, \ldots, b_q]^T$$

completely and uniqely determines the probability distribution of the process $\{y_t\}$. We will denote the set of all admissible values of θ by $\underline{\Theta}$.

The covariances of $\{y_t\}$ are defined by

$$\sigma_{yy}(n) = \sigma_{yy}(-n) = E\{y_t y_{t-n}\} \quad ; \qquad -\infty < n < \infty . \tag{2}$$

Let $S_M(\theta)$ denote the vector of M+1 consecutive sample covariances of the ARMA process whose parameter vector is θ , i.e.

$$S_M(\theta) = [\sigma_{yy}(0),\ \sigma_{yy}(1),\ldots,\sigma_{yy}(M)]^T \ . \qquad (3)$$

The sample covariance corresponding to a set of consecutive measurements $\{y_1, y_2, \ldots y_N\}$ will be defined by

$$\hat{\sigma}_{yy}(n) = \frac{1}{N-n} \sum_{t=n+1}^{N} y_t y_{t-n} \ ; \qquad 0 \leq n \leq N-1 \ . \ . \qquad (4)$$

The vector of sample covariance $\hat{S}_M$ will be defined similarly to $S_M(\theta)$, i.e.

$$\hat{S}_M = [\hat{\sigma}_{yy}(0),\ \hat{\sigma}_{yy}(1)\ .\ .\ .\ \hat{\sigma}_{yy}(M)]^T \ . \qquad (5)$$

In this paper we will consider estimates of the form

$$\hat{\theta} = g(\hat{S}_M) \ , \qquad (6)$$

where $M \geq p+q$. The function $g(\cdot)$ is assumed to satisfy the following regularity conditions:

(i) $g(\cdot)$ is continuous, with continuous partial derivatives up to a third order;

(ii) $g(\hat{S}_M)$ is a consistent estimate of θ . As is well known, $\hat{S}_M$ is a consistent estimate of $S_M(\theta)$. This, and the continuity of $g(\cdot)$ clearly imply that

$$g(S_M(\theta)) = \theta \qquad , \text{ for all } \ \theta \in \underline{\Theta} \ . \qquad (7)$$

3. THE BIAS AND THE COVARIANCE OF THE ARMA ESTIMATES

In this section we derive general asymptotic expressions for the bias and the covariance of ARMA parameter estimates of the class $g(\hat{S}_M)$ defined in the previous section. We first recall some known properties of the vector of sample covariances $\hat{S}_M$ [8, Ch. 8].

(i) $\hat{S}_M$ is unbiased,

$$E\{\hat{S}_M\} = S_M \ . \tag{8}$$

(ii) Asymptotically, the covariance of $\hat{S}_M$ is given by

$$\mathrm{Cov}\{\hat{S}_M\} = N^{-1} \textstyle\sum_M(\theta) + o(N^{-1}) \ , \tag{9}$$

where $o(N^{-1})$ denotes a term negligible compared to N^{-1}. The elements of $\sum_M(\theta)$ are given by Bartlett's formula

$$[\textstyle\sum_M(\theta)]_{k,\ell} = \sum_{m=-\infty}^{\infty} \sigma_{yy}(k+m)\sigma_{yy}(\ell+m) + \sum_{m=-\infty}^{\infty} \sigma_{yy}(k-m)\sigma_{yy}(\ell+m) \ . \tag{10}$$

An explicit expression for $\sum_M(\theta)$ was derived in [17] for the special case in which the roots of $a(z)$ are simple and appear in complex conjugate pairs. A more general formula, which holds for any ARMA process, was derived in [10] and is given in Appendix A for completeness.

(iii) The vector $\sqrt{N}(\hat{S}_M - S_M)$ is asymptotically normal with zero mean and covariance matrix $\sum_M(\theta)$.

The above properties of $\hat{S}_M$ and $g(\hat{S}_M)$ imply the following.

<u>Theorem 1:</u> Both the bias and the covariance of $\hat{\theta}$ are asymptotically proportional to N^{-1} .

<u>Proof:</u> Let $\hat{o}(N^{-1})$ denote a random variable such that

$$\lim_{N\to\infty} E\{\hat{o}(N^{-1}))^2\} = 0 \tag{11}$$

Then, since $g(\cdot)$ has continuous partial derivatives up to a third order, we can expand its k-th component $g_k(\hat{S}_M)$ in a second order Taylor series,

$$\hat{\theta}_k = g_k(\hat{S}_M) = g_k(S_M) + \frac{\partial g_k(S_M)}{\partial S_M}(\hat{S}_M - S_M)$$
$$+ \frac{1}{2}(\hat{S}_M - S_M)^T \frac{\partial^2 g_k(S_M)}{\partial S_M^2}(\hat{S}_M - S_M) + \hat{o}(N^{-1}) \quad . \tag{12}$$

Now, using the fact that $g_k(S_M) = \theta_k$ and that $\hat{S}_M$ is unbiased, we get

$$E\{\hat{\theta}_k\} - \theta_k = \frac{1}{2}E\{(\hat{S}_M - S_M)^T \frac{\partial^2 g_k(S_M)}{\partial S_M^2}(\hat{S}_M - S_M)\} + o(N^{-1})$$
$$= \frac{1}{2}\,\mathrm{tr}\{\frac{\partial^2 g_k(S_M)}{\partial S_M^2} E\{(\hat{S}_M - S_M)(\hat{S}_M - S_M)^T\}\} + o(N^{-1})$$
$$= \frac{1}{N}\frac{1}{2}\,\mathrm{tr}\{\frac{\partial^2 g_k(S_M)}{\partial S_M^2}\textstyle\sum_M(\theta)\} + o(N^{-1}) \ , \tag{13}$$

where $\mathrm{tr}\{\ \}$ denotes the trace operator. This proves that the bias of $\hat{\theta}$ is asymptotically proportional to N^{-1}. Next let us denote by $G(\theta)$ the $(p+q+1) \times (M+1)$ matrix of partial derivatives of $g(S_M)$, expressed as a function of θ. Then we get from (12),

$$\mathrm{COV}\{\hat{\theta}\} = E\{(\hat{\theta}-\theta)(\hat{\theta}-\theta)^T\}$$
$$= G(\theta)\ E\{(\hat{S}_M - S_M)(\hat{S}_M - S_M)^T\}G^T(\theta) + o(N^{-1})$$
$$= \frac{1}{N}\,G(\theta)\ \textstyle\sum_M(\theta)\ G^T(\theta) + o(N^{-1}). \tag{14}$$

This proves that the covariance of $\hat{\theta}$ is asymptotically proportional to N^{-1}.

Theorem 1 provides us with explicit asymptotic expressions for the bias and the covariance of $\hat{\theta}$ (equations (13) and (14) respectively). In particular, we observe that $\mathrm{COV}\{\hat{\theta}\}$ is asymptotically dependent only on the parameters of the given process and on the Jacobian of $g(\cdot)$. This makes eq.

(14) a useful tool in analyzing the performance of ARMA parameter estimation algorithms based on the sample covariances.

The bias formula will not be needed in the sequel. Our only interest in the bias behavior is for justifying the use of the Cramer-Rao bound for unbiased estimators, as discussed in the next section.

4. A LOWER BOUND ON THE COVARIANCE OF THE ARMA ESTIMATES

In this section we review briefly some previously published results concerning ARMA estimates based on sample covariance. While the main purpose of this section is to serve as a introduction to the next one, we have found it useful to provide alternative proofs to existing ones, due to reasons discussed in the sequel. Throughout this section, the inequality $A > B$ for matrices A and B means that A-B is positive definite. Similarly, the inequality $A \geq B$ means that A-B is a positive semidefinite matrix.

Let

$$S_M = f(\theta) \tag{15}$$

denote the functional dependence of the vector of covariances S_M on the parameter vector θ . Let $F(\theta)$ denote the $(M+1) \times (p+q+1)$ matrix of partial derivatives of $f(\cdot)$. Let $P_M(\theta)$ be the matrix

$$P_M(\theta) = [F^T(\theta) \Sigma_M^{-1}(\theta) F(\theta)]^{-1} . \tag{16}$$

<u>Theorem 2:</u>

$$G(\theta) \Sigma_M(\theta) G^T(\theta) \geq P_M(\theta) . \tag{17}$$

This theorem can be proven by observing that the matrix $P_M^{-1}(\theta)$ is, except for a factor N, the asymptotic information matrix of the sample covariances - see e.g., [17, eq. (21)]. A direct proof, which does not rely on the asymptotic normality of the sample covariances, is given as follows. The consistency requirement (7) clearly implies that

$$G(\theta)F(\theta) = I_{p+q+1} , \forall \theta \in \underline{\Theta} \tag{18}$$

when $M+1 \geq p+q+1$. Also,

$$
\begin{array}{r l}
M+1 \; \{ & \multirow{2}{*}{$\begin{bmatrix} \Sigma_M(\theta) & 0 \\ 0 & 0 \end{bmatrix}$} \\
p+q+1 \; \{ & \\
& \;\;\underbrace{}_{M+1}\;\underbrace{}_{p+q+1}
\end{array} > 0 \tag{19}
$$

Hence we have (omitting the dependence on θ for convenience)

$$
\begin{bmatrix} G & 0 \\ F^T\Sigma_M^{-1} & F^T \end{bmatrix}
\begin{bmatrix} \Sigma_M & 0 \\ 0 & 0 \end{bmatrix}
\begin{bmatrix} G^T & \Sigma_M^{-1}F \\ 0 & F \end{bmatrix}
=
\begin{bmatrix} G\Sigma_M G^T & GF \\ F^T G^T & F^T\Sigma_M^{-1}F \end{bmatrix}
=
$$

$$
= \begin{bmatrix} G\Sigma_M G^T & I_{p+q+1} \\ I_{p+q+1} & F^T\Sigma_M^{-1}F \end{bmatrix} > 0 \; . \tag{20}
$$

Therefore,

$$
\begin{bmatrix} I_{p+q+1} & -(F^T\Sigma_M^{-1}F)^{-1} \\ 0 & I_{p+q+1} \end{bmatrix}
\begin{bmatrix} G\Sigma_M G^T & I_{p+q+1} \\ I_{p+q+1} & F^T\Sigma_M^{-1}F \end{bmatrix}
\begin{bmatrix} I_{p+q+1} & 0 \\ -(F^T\Sigma_M^{-1}F)^{-1} & I_{p+q+1} \end{bmatrix}
$$

$$
= \begin{bmatrix} G\Sigma_M G^T-(F^T\Sigma_M^{-1}F)^{-1} & 0 \\ 0 & F^T\Sigma_M^{-1}F \end{bmatrix} > 0 \; , \tag{21}
$$

and finally

$$
G(\theta)\Sigma_M(\theta)G^T(\theta) > [F^T(\theta)\Sigma_M^{-1}(\theta)F(\theta)]^{-1} \; . \tag{22}
$$

By theorem 2, $N^{-1}P_M(\theta)$ is an asymptotic lower bound on the covariance of any estimate $\hat{\theta} = g(\hat{S}_M)$. The closed-form expressions for $\Sigma_M(\theta)$ and $F(\theta)$ given in appendix A enable the computation of this bound as a function of the ARMA process parameters θ , without computing the roots of $a(z)$ as was required in [17].

We now turn our attention to the relationship between the bound given in (17) and the Cramer-Rao bound (CRB). For a biased estimate $\hat{\theta}$ with a bias term $b(\theta)$ the CRB is given by [11, Ch. 4]

$$\mathrm{COV}\{\hat{\theta}\} > [I_{p+q+1} + \frac{\partial b(\theta)}{\partial \theta}] I_N^{-1}(\theta) [I_{p+q+1} + \frac{\partial b(\theta)}{\partial \theta}]^T , \tag{23}$$

where $I_N(\theta)$ is the Fisher information matrix corresponding to θ (assuming N measurements) and $\frac{\partial b(\theta)}{\partial \theta}$ is the Jacobian matrix of $b(\theta)$. From theorem 1 we know that $b(\theta)$ is asymptotically proportional to N^{-1}. Hence the same is true for $\frac{\partial b(\theta)}{\partial \theta}$. Also, $I_N(\theta)$ is known to be asymptotically proportional to N [1]. Therefore, the right-hand side of (23) differs from $I_N^{-1}(\theta)$ by a term $o(N^{-1})$, and we can asymptotically replace (23) by

$$\mathrm{COV}\{\hat{\theta}\} + O(N^{-1}) > I_N^{-1}(\theta) = \mathrm{CRB}\{\theta\} \ . \tag{24}$$

This argument justifies the comparison of the bound $P_M(\theta)$ to $I_N^{-1}(\theta)$ rather than to the bound given in (23) in the asymptotic case. The next theorem asserts that, for every finite M, $P_M(\theta)$ is strictly greater than the CRB.

<u>Theorem 3:</u>

$$P_M(\theta) > N\, I_N^{-1}(\theta) \tag{25}$$

This theorem follows as a special case of a more general theorem - see Theorem 5 and the corollory preceding it in [16]. An alternative proof follows from Theorem 4 in [10]. There is was shown that for all $M > p-q$,

$$\Sigma_M(\theta) > N\, F(\theta) I_N^{-1}(\theta) F^T(\theta) \ . \tag{26}$$

Therefore we have (omitting the dependence on θ for convenience),

$$[(F^T \Sigma_M^{-1} F)^{-1} F^T \Sigma_M^{-1}]\, \Sigma_M [\Sigma_M^{-1} F (F^T \Sigma_M^{-1} F)^{-1}]$$

$$> N\, [(F^T \Sigma_M^{-1} F)^{-1} F^T \Sigma_M^{-1}][F I_N^{-1} F^T][\Sigma_M^{-1} F (F^T \Sigma_M^{-1} F)^{-1}] \ . \tag{27}$$

Hence

$$(F^T \Sigma_M^{-1} F)^{-1} > N\ I_N^{-1} . \tag{28}$$

Note that the condition $M > p-q$ implies $M > p+q$ for all $q \geq 1$. The only exception is $q=0$, i.e., the case of a pure AR process. Indeed, for AR processes it can be shown (using the formulas given in [10]) that

$$(F^T \Sigma_p^{-1} F)^{-1} = N I_N^{-1} , \tag{29}$$

i.e., estimates of the AR parameters based on the sample covariances $\hat{S}_p$ (e.g., the Yule-Walker estimate) can be asymptotically efficient.

It is not difficult to show that the bound $P_M(\theta)$ is tight, i.e., there exists an estimate $\hat{\theta}(\hat{S}_M)$ that asymptotically achieves this bound. To show this, let us define

$$V(X,\hat{S}_M) = [S_M(x) - \hat{S}_M]^T\ \Sigma_M^{-1}(x)[S_M(x) - \hat{S}_M] \qquad , x \in \underline{\Theta} . \tag{30}$$

We define $\hat{\theta}(\hat{S}_M)$ to be the value of $x \in \underline{\Theta}$ for which $V(x,\hat{S}_M)$ attains a global minimum. The estimate $\hat{\theta}$ satisfies the consistency condition (7) since clearly

$$V(\theta,\ S_M(\theta)) = 0 , \tag{31}$$

i.e., the true parameter θ is a global minimizer of V when the sample covariances $\hat{S}_M$ are replaced by the true covariances $S_M(\theta)$. Also, $V(x,\hat{S}_M)$ is a rational function of x and $\Sigma_M(x)$ is nonsigular for all $x \in \underline{\Theta}$.† Hence the partial derivatives of $g(\cdot)$ exist and are continuous to any order. The following theorem asserts that the estimate $\hat{\theta}$ defined above achieves the bound $P_M(\theta)$.

†Note that $\Sigma_M(x)$ is a rational function of x, as shown in the appendix.

Theorem 4: Asymptotic covariance matrix of the estimate $\hat{\theta}$ defined above is given by

$$\lim_{N\to\infty} N \operatorname{Cov}\{\hat{\theta}\} = P_M(\theta) . \tag{32}$$

Proof:

By eq. (14), we only have to show that the Jacobian $G(\theta)$ yields equality in (17). The first step is to show that the Jacobian is given by

$$G(\theta) = -\left[\left. \frac{\partial^2 V(X,\hat{S}_M)}{\partial X^2} \right|_{\substack{x=\theta \\ \hat{S}_M=S_M}} \right]^{-1} \left[\left. \frac{\partial^2 V(x,\hat{S}_M)}{\partial X \partial \hat{S}_M} \right|_{\substack{x=\theta \\ \hat{S}_M=S_M}} \right] . \tag{33}$$

The next step is to evaluate the two terms at the right-hand side of (33). This yields

$$\left. \frac{\partial^2 V(x,\hat{S}_M)}{\partial X^2} \right|_{\substack{x=\theta \\ S_M=S_M}} = 2F^T(\theta) \Sigma_N^{-1}(\theta) F(\theta) \tag{34a}$$

$$\left. \frac{\partial^2 V(X,\hat{S}_M)}{\partial X^2} \right|_{\substack{x=\theta \\ S_M=S_M}} = -2F^T(\theta) \Sigma_M^{-1}(\theta) \tag{34b}$$

$$G(\theta) = [F^T(\theta)\Sigma_M^{-1}(\theta)F(\theta)]^{-1} F^T(\theta)\Sigma_M^{-1}(\theta) . \tag{34c}$$

Finally, when $G(\theta)$ is substituted into (14), we obtain the stated equality (32). See appendix B for a more complete proof.

An algorithm of the type discussed here was given in [20], and is closely related to the one proposed by Walker [5]. This algorithm requires a considerable amount of computations, due to the need to invert $\Sigma_M(\theta)$ at each iteration. Thus, it is not necessarily recommended for practical applications.

5. THE LIMITING BEHAVIOR OF $P_M(\theta)$

So far we have restricted our discussion to estimates based on a fixed number of sample covariances. In this section we examine the limiting behavior of the lower bound $P_M(\theta)$ as the number of sample covariances M goes to infinity.

We will show that the limit of $P_M(\theta)$ is equal to the asymptotic Cramér-Rao bound. Therefore, the relative asymptotic efficiency of estimates of the form $\hat{\theta} = g(\hat{S}_M)$ can be made arbitrarily close to unity by increasing the number of sample covariances and by using them in an optimal manner (e.g., as in the estimate of Theorem 4).

Let us denote

$$I(\theta) = \lim_{N\to\infty} N^{-1} I_N(\theta) \ , \tag{35}$$

i.e., $I(\theta)$ is the limiting information matrix, normalized by the number of data points. Let $\phi(\omega)$ denote the power spectral density of the given process, i.e.,

$$\phi(\omega) = \sum_{m=-\infty}^{\infty} \sigma_{yy}(m) \cos m\omega \ . \ . \tag{36}$$

For ARMA processes, the power spectral density is an absolutely continuous function of ω. Also, since b(z) was assumed to be nonzero on the unit circle, $\phi(\omega)$ is strictly positive for all ω. Therefore, the elements of $I(\theta)$ can be expressed by Whittle's asymptotic formula [20]

$$I_{k,\ell}(\theta) = \frac{1}{4\pi}\int_{-\pi}^{\pi} \frac{1}{\phi^2(\omega)} \frac{\partial\phi(\omega)}{\partial\theta_k} \frac{\partial\phi(\omega)}{\partial\theta_\ell} d\omega \ , \quad 1 \le k,\ell \le p+q+1 \ . \tag{37}$$

Using this formula, we will now prove the following theorem.

<u>Theorem 5:</u>

$$\lim_{M\to\infty} P_M^{-1}(\theta) = I(\theta) \ . \tag{38}$$

Proof:

Let $L_2[-\pi,\pi]$ be the Hilbert space of Lebesgue measurable functions on $[-\pi,\pi]$ whose square-magnitudes are Lebesgue integrable. Let H denote the subspace of $L_2[-\pi,\pi]$ consisting of all real even functions. The inner product of two members of H is given by

$$\langle h_1(\omega),\ h_2(\omega)\rangle = \int_{-\pi}^{\pi} h_1(\omega) h_2(\omega) d\omega \ . \tag{38}$$

For ARMA processes, the spectral density $\phi(\omega)$ satisfies

$$0 < \beta_1 \leq \phi(\omega) \leq \beta_2 < \infty \ , \quad -\pi \leq \omega \leq \pi \ . \tag{39}$$

The functions $\phi(\omega)$, $\frac{\partial\phi(\omega)}{\partial\theta_k}$ and $\frac{1}{\phi(\omega)} \cdot \frac{\partial\phi(\omega)}{\partial\theta_k}$ all belong to H. Let us define

$$v_k(\omega) = \frac{\phi(\omega)\cos k\omega}{\sqrt{\pi}} \ ; \quad k = 0,1,2,\ldots \ . \tag{40}$$

The sequence $\{v_k(\omega),\ k \geq 0\}$ spans the space H. To see this, suppose that there exists $h(\omega)\varepsilon H$ such that

$$\int_{-\pi}^{\pi} h(\omega) \cdot \frac{\phi(\omega)\cos k\omega}{\sqrt{\pi}} d\omega = 0, \qquad \forall_k \tag{41}$$

Since $\phi(\omega)$ is bounded, $h(\omega)\phi(\omega)\varepsilon H$. Furthermore, the sequence $\{\cos k\omega,\ k \geq 0\}$ is known to be complete in H. Hence $\phi(\omega)h(\omega) \equiv 0$, and since $\phi(\omega)$ is srictly positive, $h(\omega) \equiv 0$. This proves that $\{v_k(\omega),\ k \geq 0\}$ spans H.

Let $\{u_k(\omega),\ k \geq 0\}$ be the sequence obtained by Gram-Schmidt orthonormalization of $\{v_k(\omega),\ k \geq 0\}$. From the discussion above it follows that $\{u_k(\omega),\ k \geq 0\}$ is a complete orthonormal sequence. The two sequences are known to be related via

$$\begin{bmatrix} u_0(\omega) \\ u_1(\omega) \\ \vdots \\ u_M(\omega) \end{bmatrix} = [W_M^{1/2}]^{-1} \begin{bmatrix} v_0(\omega) \\ v_1(\omega) \\ \vdots \\ v_M(\omega) \end{bmatrix} , \quad \forall\ M \geq 0 \tag{42}$$

where W_M is the Gramian matrix of $\{v_k(\omega),\ 0 \le k \le M\}$ and $W_M^{1/2}$ is its lower triangular square-root. Recall that the Gramian matrix is given by

$$(W_M)_{k,\ell} = \langle v_k(\omega),\ v_\ell(\omega)\rangle \ . \tag{43}$$

Next note that (see e.g. [8, p. 465, Theorem 8.3.3])

$$[\Sigma_M(\theta)]_{k,\ell} = \lim_{N\to\infty} N \cdot \mathrm{cov}\{\hat{\sigma}_{yy}(k),\ \hat{\sigma}_{yy}(\ell)\}$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} \phi^2(\omega)\cos k\omega \cos \ell\omega\, d\omega = \int_{-\pi}^{\pi} v_k(\omega) v_\ell(\omega)\, d\omega = \langle v_k(\omega),\ v_\ell(\omega)\rangle \ . \tag{44}$$

Hence the Gramian in (42) is just $\Sigma_M(\theta)$, i.e.,

$$W_M = \Sigma_M(\theta) \qquad , M > 0. \tag{45}$$

Let us now turn our attention to the entries of the matrix F. We have

$$\frac{\partial\sigma_{yy}(n)}{\partial\theta_k} = \frac{\partial}{\partial\theta_k}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\phi(\omega)\cos n\omega\, d\omega\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\partial\phi(\omega)}{\partial\theta_k}\cos n\omega\, d\omega$$

$$= \int_{-\pi}^{\pi}\left[\frac{1}{2\sqrt{\pi}\ \phi(\omega)} \cdot \frac{\partial\phi(\omega)}{\partial\theta_k}\right]\cdot\left[\frac{\phi(\omega)\cos n\omega}{\sqrt{\pi}}\right] d\omega = \langle\alpha(\omega),\ v_n(\omega)\rangle \ , \tag{46}$$

where

$$\alpha(\omega) = \frac{1}{2\sqrt{\pi}\ \phi(\omega)} \cdot \frac{\partial\phi(\omega)}{\partial\theta_k} \ . \tag{47}$$

Similary,

$$\frac{\partial\sigma_{yy}(n)}{\partial\theta_\ell} = \langle\beta(\omega),\ v_n(\omega)\rangle \ , \tag{48}$$

where

$$\beta(\omega) = \frac{1}{2\sqrt{\pi}\ \phi(\omega)} \cdot \frac{\partial\phi(\omega)}{\partial\theta_\ell} \ . \tag{49}$$

Using (16), (42), (45), (46) and (48) we see that

$$[P_M^{-1}(\theta)]_{k,\ell} = \sum_{m=0}^{M} \sum_{n=0}^{M} \langle\alpha(\omega), v_m(\omega)\rangle [\Sigma_M^{-1}(\theta)]_{m,n} \langle\beta(\omega), v_n(\omega)\rangle$$

$$= \sum_{m=0}^{M} \langle\alpha(\omega), u_m(\omega)\rangle\langle\beta(\omega), u_m(\omega)\rangle = \langle\alpha_M(\omega), \beta_M(\omega)] \ . \qquad (50)$$

where $\alpha_M(\omega)$ and $\beta_M(\omega)$ are, respectively, the projections of $\alpha(\omega)$ and $\beta(\omega)$ on the subspace spanned by $\{u_m(\omega),\ 0 < m < M\}$.

By Whittle's formula (37) we have

$$I_{k,\ell}(\theta) = \int_{-\pi}^{\ell} \left[\frac{1}{2\sqrt{\pi}\ \phi(\omega)} \cdot \frac{\partial\phi(\omega)}{\partial\theta_k}\right] \cdot \left[\frac{1}{2\sqrt{\pi}\ \phi(\omega)} \cdot \frac{\partial\phi(\omega)}{\partial\theta_\ell}\right] d\omega = \langle\alpha(\omega),\ \beta(\omega)\rangle \qquad (51)$$

Thus, it only remains to show that

$$\lim_{M\to\infty} \langle\alpha_M(\omega),\ \beta_M(\omega)\rangle = \langle\alpha(\omega),\ \beta(\omega)\rangle \ . \qquad (52)$$

By the completeness of $\{u(\omega),\ m > 0\}$ we have (omitting the dependence on ω for convenience)

$$|\langle\alpha_M,\ \beta_M\rangle - \langle\alpha,\ \beta\rangle| < |\langle\alpha_M, \beta_M\rangle - \langle\alpha_M,\ \beta\rangle|$$

$$+ |\langle\alpha_M,\ \beta\rangle - \langle\alpha,\ \beta\rangle| = |\langle\alpha_M,\ \beta_M - \beta\rangle| + |\langle\alpha_M - \alpha,\ \beta\rangle|$$

$$< \|\alpha_M\| \cdot \|\beta_M - \beta\rangle| + \|\alpha_M - \alpha\| \cdot \|\beta\| \xrightarrow{M \to \infty} 0 \ . \qquad (53)$$

This completes the proof of the theorem.

<u>Remark:</u> It is clear from the proof that the theorem is not restricted to ARMA processes. In fact, the following conditions are sufficient for the theorem to hold.

(i) The process power spectral density $\phi(\omega)$ of the process satisfies (39).

(ii) The partial derivatives $\partial\phi(\omega)/\partial\theta_k$ belong to $L_2[-\pi,\pi]$.

(iii The partial derivatives are sufficiently regular to allow interchange of the differentiation and integration in (46).

6. SOME NUMERICAL EXAMPLES

In this section we illustrate the behavior of the bound (16) as a function of the number of sample covariances M, by two examples. In both examples the ARMA processes are of order (2,2), with a pair of complex zeroes and a pair of complex poles. the zeroes are at angles ±45° with respect to the positive real axis, and the poles are at angles ±135°. The absolute values of both the zeroes and the poles are $(0.5)^{-1/2}$ in the first example and $(0.9)^{-1/2}$ in the second. The corresponding polynomials are

$$\frac{b(z)}{a(z)} = \frac{1-z+0.5z^2}{1+z+0.5z^2} \qquad \text{(first example)} \qquad (54a)$$

$$\frac{b(z)}{a(z)} = \frac{1-1.273z+0.9z^2}{1+1.273z+0.9z^2} \qquad \text{(second example)} \qquad (54b)$$

In the first example we computed the bound up to M=20. The bounds on the standard deviations of $\hat{a}_1$, $\hat{a}_2$, $\hat{b}_1$, $\hat{b}_2$ are shown in figures 1a, 1b, 1c, 1d respectively. Also shown in the figures are the respective Cramer-Rao bounds (standard deviations) on the parameters (the horizontal lines). As we see, for $M \geq 12$ the bound is practically indistinguishable from the CRB. For M=4, the minimum possible number, any ARMA method based on sample covariances would be quite inefficient.

In the second example we computed the bound up to M=50. The results are shown in figures 2a, 2b, 2c, 2d. As we see, the effect of moving the poles towards the unit circle is to enhance the relative efficiency of the AR coefficients. On the other hand, moving the zeroes towards the unit circle results in a slow decrease of the bound on the MA coefficients. In this example, even at M=50 there is still a considerable gap between the bound in $\hat{b}_1$ and $\hat{b}_2$ and the respective CRB's.

7. CONCLUSION

In this paper we presented expressions for the asymptotic accuracy of ARMA parameter estimation techniques based on a finite number of sample covariances. The error covariance matrix of any estimation technique of this class is bounded from below by a bound which is strictly larger than the Cramer-Rao bound. Furthermore, this lower bound is tight: it can be achieved by using the specific ARMA estimation method given in section 4. It was also shown that this lower bound approaches the Cramer-Rao bound as the number of sample covariances tends to infinity.

Finally we remark that the results presented in sections 3 and 4 can be easily generalized to the situation where only a subset of $\{\hat{\sigma}_{yy}(0),\ldots,\hat{\sigma}_{yy}(M)\}$ is used. All that needs to be done is to delete appropriate rows and columns from the matrices $\Sigma_M(\theta)$, $G(\theta)$ and $F(\theta)$. The variance expressions remain otherwise unchanged. However, the results of section 5 will no longer be valid: discarding some of the sample covariance will generally cause loss of efficiency. See [15],[17],[18] for the case where the set of sample covariances starts at the k-th lag, rather than at the zero-th lag.

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APPENDIX A: EXPLICIT EXPRESSIONS FOR $F(\theta)$ AND $\Sigma_M(\theta)$

In this appendix we quote, without proofs, some results derived in [10]. Let us first introduce some notations as follows. Let $\{r_{xx}(\ell)\}$, $\{r_{xw}(\ell)\}$ and $\{r_{yy}(\ell)\}$ be the coefficients in the Laurent series

$$\frac{1}{a(z)a(z^{-1})} = \sum_{\ell=-\infty}^{\infty} r_{xx}(\ell)z^{\ell} \; ; \; \frac{1}{a(z)b(z^{-1})} = \sum_{\ell=-\infty}^{\infty} r_{xw}(\ell)z^{\ell} \; ;$$

$$\frac{b(z)b(z^{-1})}{a(z)a(z^{-1})} = \sum_{\ell=-\infty}^{\infty} r_{yy}z^{\ell} \; . \tag{A1}$$

Let $R_{xx}^{i,j}(\ell)$, $R_{xw}^{i,j}(\ell)$ and $R_{yy}^{i,j}(\ell)$ be the ixj Toeplitz matrices

$$\{R_{xx}^{i,j}(\ell)\}_{m,n} = r_{xx}(\ell-m+n) \; ; \; \{R_{xw}^{i,j}(\ell)\}_{m,n} = r_{xw}(\ell-m+n). \; ;$$

$$\{R_{yy}^{i,j}(\ell)\}_{m,n} = r_{yy}(\ell-m+n). \tag{A2}$$

Let A and B be the pxp and qxq companion matrices

$$\{A\}_{m,n} = \begin{cases} -a_k & ; \; m=1, n=k \\ 1 & ; \; m=n+1 \\ 0 & ; \; \text{otherwise} \end{cases} \; ; \; \{B\}_{m,n} = \begin{cases} -b_k & ; \; m=1, \; n=k \\ 1 & ; \; m=n+1 \\ 0 & ; \; \text{otherwise} \end{cases} \; . \tag{A3}$$

Let H be the (M+q) x (M+q) Hankel matrix

$$\{H\}_{m,n} = \begin{cases} 1 & ; \; m+n = M+q+1 \\ a_k & ; \; m+n = M+q+k+1 \; , \quad 1 \leq k \leq p \; . \\ 0 & ; \; \text{otherwise} \end{cases} \tag{A4}$$

Let J be the (M+1)x(M+1) matrix

$$\{J\}_{m,n} = \begin{cases} 1 & ; \; m+n = M+2 \\ 0 & ; \; \text{otherwise} \end{cases} \; . \tag{A5}$$

Let e_i be an i-dimensional unit vector with 1 in the first position and zeros elsewhere.

Finally, let K be the (M+q)x(M+1) matrix defined by

$$K = H\{R_{yy}^{M+q,M+1}(M+q) + R_{yy}^{M+q,M+1}(q)J\} \quad . \tag{A6}$$

Then we have the following results.

Lemma 1: The matrices $R_{xx}^{p,p}(0)$ and $R_{xw}^{p,q}(0)$ satisfy the matrix Lyapunov equations

$$R_{xx}^{p,p}(0) - AR_{xx}^{p,p}(0)A^T = e_p e_p^T \tag{A7}$$

$$R_{xw}^{p,q}(0) - AR_{xw}^{p,q}(0)B^T = e_p e_q^T \tag{A8}$$

These Lyapunov equations admit rational closed-form solutions as explained, e.g., in [12].

Lemma 2: $\{r_{xx}(\ell)\}$ and $\{r_{xw}(\ell)\}$ can be computed for all $\ell > p$ using the solutions to (A7), (A8), and the recursions

$$r_{xx}(\ell) = \sum_{k=1}^{p} a_k r_{xx}(\ell-k) \; ; \quad r_{xw}(\ell) = - \sum_{k=1}^{p} a_k r_{xw}(\ell-k) \; ; \; \ell > p \; . \tag{A9}$$

Hence, the matrices $R_{xx}^{i,j}(\ell)$ and $R_{xw}^{i,j}(\ell)$ can be computed for any desired values of i,j and ℓ .

Lemma 3: Denote by $\nabla_a S_M$ the matrix of partial derivatives of S_M with respect to $\{a_1, a_2, \ldots a_p\}$, and similarly for $\nabla_b S_M$. Then

$$\nabla_a S_M = -\sigma_u^2 K^T R_{xx}^{M+q,p}(0) \tag{A10}$$

$$\nabla_b S_M = +\sigma_u^2 \, K^T R_{xw}^{M+q,q}(0) \tag{A11}$$

$$\frac{\partial S_M}{\partial \sigma_u^2} = \frac{1}{\sigma_u^2} S_M \tag{A12}$$

Then the matrix $F(\theta)$ is given by

$$F(\theta) = [\frac{\partial S_M}{\partial \sigma_u^2} \vdots \nabla_a S_M \vdots \nabla_b S_M] \tag{A13}$$

<u>Lemma 4:</u> The matrix $\Sigma_M(\theta)$ is given by

$$\Sigma_M(\theta) = 2S_M S_M^T + \sigma_u^4 K^T R_{xx}^{M+q,M+q}(0)K . \tag{A14}$$

APPENDIX B: PROOF OF THEOREM 4

It is sufficient to evaluate $G(\hat{\theta})$ and then compute the corresponding value of the right-hand side of (14). Since $\hat{\theta}$ is a global minimizer of $V(x,\hat{S})$,

$$\left.\frac{\partial V(x,S_M)}{\partial x}\right|_{x=\hat{\theta}} = 0 . \tag{B1}$$

Let us now perturb $\hat{S}_M$ by a differential amount $d\hat{S}_M$, and let $\hat{\theta} + d\hat{\theta}$ be the global minimizer of $V(x,\hat{S}_M + d\hat{S}_M)$. Then

$$\left.\frac{\partial V(x,\hat{S}_M + d\hat{S}_M)}{\partial x}\right|_{x=\hat{\theta}+d\hat{\theta}} = 0 . \tag{B2}$$

$$\left.\frac{\partial V(x,\hat{S}_M+d\hat{S}_M)}{\partial x}\right|_{x=\hat{\theta}+d\hat{\theta}} = \left.\frac{\partial V(x,\hat{S}_M)}{\partial x}\right|_{x=\hat{\theta}}$$

$$+ \left.\frac{\partial^2 V(x,\hat{S}_M)}{\partial x^2}\right|_{x=\hat{\theta}} d\hat{\theta} + \left.\frac{\partial^2 V(x,\hat{S}_M)}{\partial x \partial \hat{S}_M}\right|_{x=\hat{\theta}} d\hat{S}_M . \tag{B3}$$

Using (B1) and (B2) we get

$$d\hat{\theta} = -\left[\left.\frac{\partial^2 V(x,\hat{S}_M)}{\partial x^2}\right|_{x=\hat{\theta}}\right]^{-1} \left[\left.\frac{\partial^2 V(x,\hat{S}_M)}{\partial x \partial \hat{S}_M}\right|_{x=\hat{\theta}}\right] d\hat{S}_M . \tag{B4}$$

Therefore, the matrix of partial derivatives of $g(\cdot)$ is given by

$$G(\hat{\theta}) = -\left[\left.\frac{\partial^2 V(x,\hat{S}_M)}{\partial x^2}\right|_{x=\hat{\theta}}\right]^{-1} \left[\left.\frac{\partial^2 V(x,\hat{S}_M)}{\partial x \partial \hat{S}_M}\right|_{x=\hat{\theta}}\right] . \tag{B5}$$

Next we evaluate the partial derivatives appearing in (B5)

$$\frac{\partial V}{\partial x_k} = [\frac{\partial S_M(x)}{\partial x_k}]^T \Sigma_M^{-1}(x)[S_M(x) - \hat{S}_M] + [S_M(x)-\hat{S}_M]^T \Sigma_M^{-1}(x)[\frac{\partial S_M(x)}{\partial x_k}]$$

$$-[S_M(x)-\hat{S}_M]^T \Sigma_M^{-1}(x) \frac{\partial \Sigma_M(x)}{\partial x_k} \Sigma_M^{-1}(x)[S_M(x)-\hat{S}_M] \tag{B6}$$

$$\frac{\partial^2 V}{\partial x_k \partial x_\ell} = [\frac{\partial^2 S_M(x)}{\partial x_k \partial x_\ell}] \Sigma_M^{-1}(x)[S_M(x)-\hat{S}_M]$$

$$-[\frac{\partial S_M(x)}{\partial x_k}]^T \Sigma_M^{-1}(x) \frac{\partial \Sigma_M(x)}{\partial x_\ell} \Sigma_M^{-1}(x)[S_M(x)-\hat{S}_M]$$

$$+ [\frac{\partial S_M(x)}{\partial x_k}]^T \Sigma_M^{-1}(x)[\frac{\partial S_M(x)}{\partial x_\ell}] \qquad + [\frac{\partial S_M(x)}{\partial x_\ell}]^T \Sigma_M^{-1}(x)[\frac{\partial S_M(x)}{\partial x_k}]$$

$$-[S_M(x)-\hat{S}_M]^T \Sigma_M^{-1}(x) \frac{\partial \Sigma_M(x)}{\partial x_\ell} \Sigma_M^{-1}(x)[\frac{\partial S_M(x)}{\partial x_k}] + [S_M(x)-\hat{S}_M]^T \Sigma_M^{-1}(x)[\frac{\partial^2 S_M(x)}{\partial x_k \partial x_\ell}]$$

$$-[\frac{\partial S_M(x)}{\partial x_\ell}]^T \Sigma_M^{-1}(x) \frac{\partial \Sigma_M(x)}{\partial x_k} \Sigma_M^{-1}(x)[S_M(x)-\hat{S}_M]$$

$$+ [S_M(x)-\hat{S}_M]^T \Sigma_M^{-1}(x) \frac{\partial \Sigma_M(x)}{\partial x_\ell} \Sigma_M^{-1}(x) \frac{\partial \Sigma_M(x)}{\partial x_k} \Sigma_M^{-1}(x)[S_M(x)-\hat{S}_M]$$

$$-[S_M(x)-\hat{S}_M]^T \Sigma_M^{-1}(x) \frac{\partial^2 \Sigma_M(x)}{\partial x_k \partial x_\ell} \Sigma_M^{-1}(x)[S_M(x)-\hat{S}_M]$$

$$+[S_M(x)-\hat{S}_M]^T \Sigma_M^{-1}(x) \frac{\partial \Sigma_M(x)}{\partial x_k} \Sigma_M^{-1}(x) \frac{\partial \Sigma_M(x)}{\partial x_\ell} \Sigma_M^{-1}(x)[S_M(x)-\hat{S}_M]$$

$$-[S_M(x)-\hat{S}_M]^T \Sigma_M^{-1}(x) \frac{\partial \Sigma_M(x)}{\partial x_k} \Sigma_M^{-1}(x)[\frac{\partial S_M(x)}{\partial x_\ell}] \tag{B7}$$

$$\frac{\partial^2 V}{\partial x_k \partial \hat{S}_M} = -2\left[\frac{\partial S_M(x)}{\partial x_k}\right]^T \Sigma_M^{-1}(x) + 2\Sigma_M^{-1}(x)\, \frac{\partial \Sigma_M(x)}{\partial x_k}\, \Sigma_M^{-1}(x)[S_M(x)-\hat{S}_M] \tag{B8}$$

Recall now that in (14), $G(\hat{\theta})$ has to evaluated at the <u>true</u> values of $\hat{\theta}$ and $\hat{S}_M$, i.e. at $\hat{\theta} = \theta$ and $\hat{S}_M = S_M(\theta)$. Substituting in (B7) and (B8) we see that most of the terms vanish and we get

$$\left.\frac{\partial^2 V}{\partial x^2}\right|_{x=\theta} = 2\left[\frac{\partial S_M(x)}{\partial x}\right]^T_{x=\theta} \Sigma_M^{-1}(\theta)\left[\frac{\partial S_M(x)}{\partial x}\right]_{x=\theta}$$

$$= 2F^T(\theta)\, \Sigma_M^{-1}(\theta)F(\theta) \tag{B9}$$

$$\left.\frac{\partial^2 V}{\partial x \partial \hat{S}_M}\right|_{x=\theta} = -2\left[\frac{\partial S_M(x)}{\partial x}\right]^T_{x=\theta} \Sigma_M^{-1}(\theta) = -2F^T(\theta)\Sigma_M^{-1}(\theta) \ . \tag{B10}$$

Finally we get,

$$G(\theta) = (F^T\Sigma_M^{-1}F)^{-1}F^T\Sigma_M \tag{B11}$$

and

$$G(\theta)\Sigma_M(\theta)G^T(\theta) = (F^T\Sigma_M^{-1}F)^{-1}F^T\Sigma_M^{-1}\Sigma_M\Sigma_M^{-1}F(F^T\Sigma_M^{-1}F)^{-1}$$

$$= (F^T\Sigma_M{}^{-1}F)^{-1}(F^T\Sigma_M^{-1}F)(F^T\Sigma_M^{-1}F)^{-1}$$

$$= [F^T(\theta)\Sigma_M^{-1}(\theta)F(\theta)]^{-1}. \tag{B12}$$

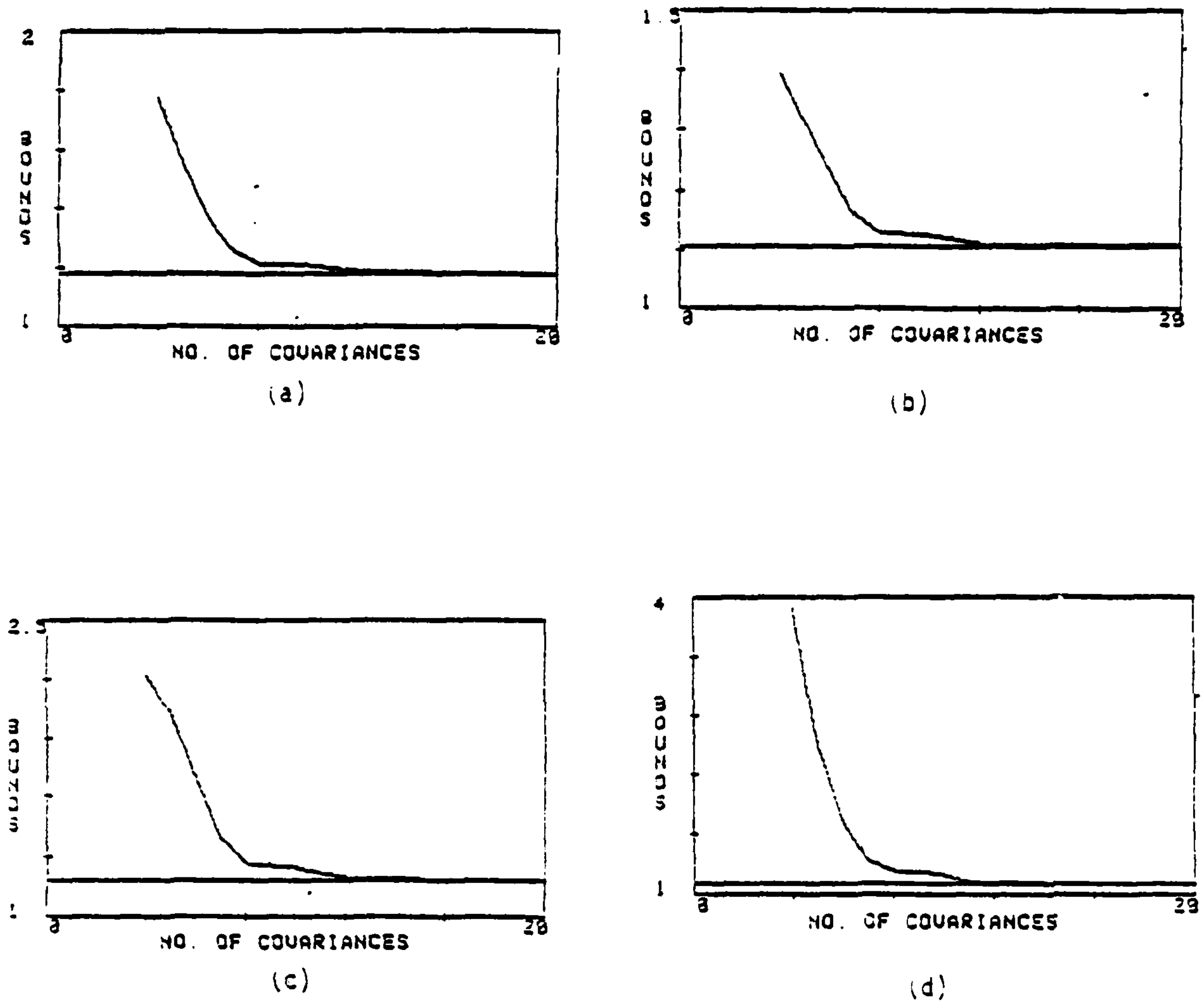

Figure 1: The Bounds on the ARMA Parameters in Example 1:
a) $\hat{a}_1$, b) $\hat{a}_2$, c) $\hat{b}_1$, d) $\hat{b}_2$

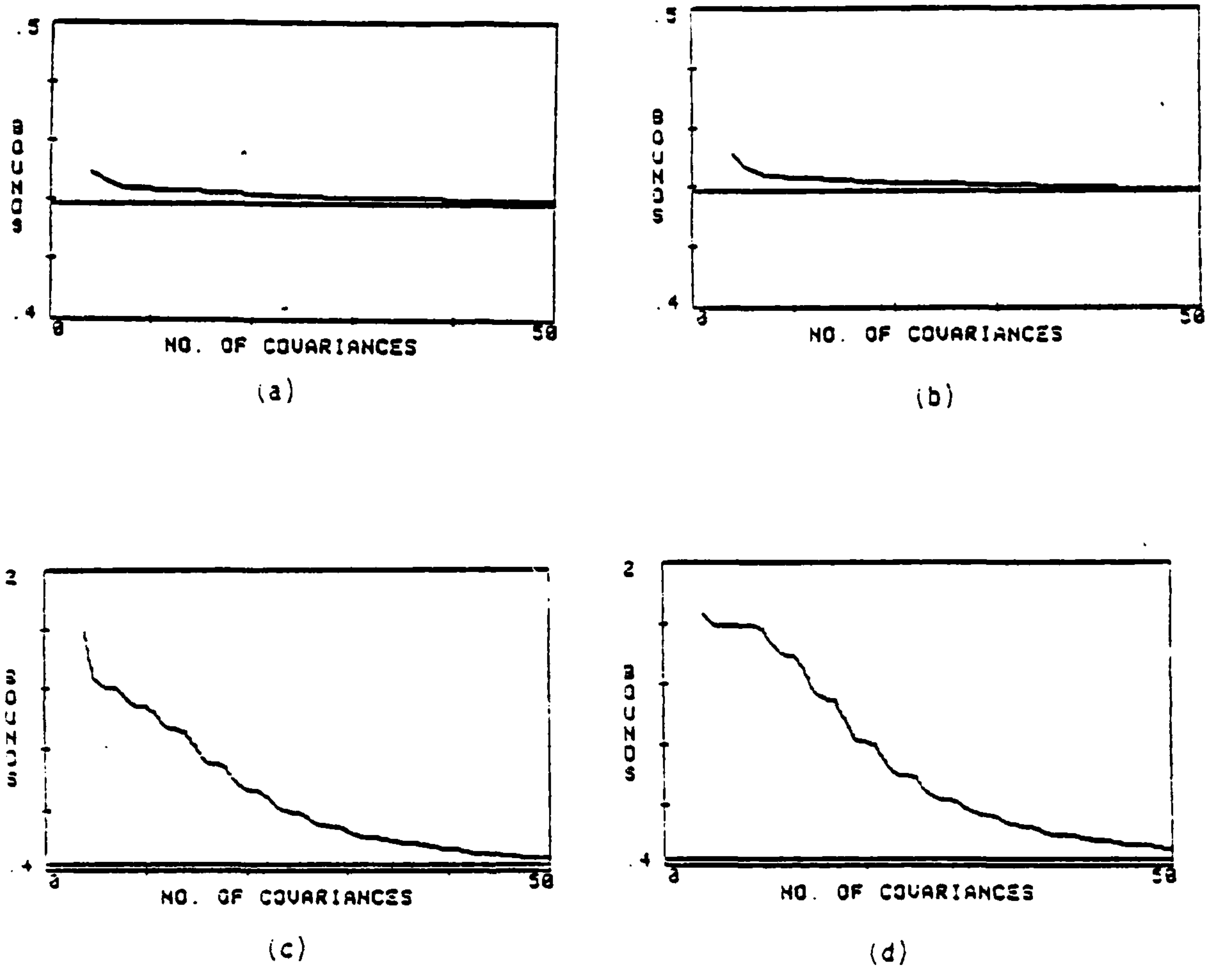

Figure 2: The Bounds on the ARMA Parameters in Example 2:
a) $\hat{a}_1$, b) $\hat{a}_2$, c) $\hat{b}_1$, d) $\hat{b}_2$

APPENDIX K

THE EXACT CRAMER-RAO BOUND FOR GAUSSIAN AUTOREGRESSIVE PROCESSES

THE EXACT CRAMER-RAO BOUND FOR GAUSSIAN AUTOREGRESSIVE PROCESSES

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ABSTRACT

An explicit expression is derived for the Cramer-Rao bound on unbiased estimates of the parameters of autoregressive processes, given a finite number of measurements. The expression converges to the well-known asymptotic form of the CRB when the number of measurements tends to infinity. The behavior of the bound is ilustrated by some numerical examples.

This work was supported by the Army Research Office under contract no. DAAG29-83-C-0027.

1. INTRODUCTION

Autoregressive (AR) modeling techniques are widely used for spectral analysis, estimation/prediction of stationary time series, and adaptive filtering. Numerous algorithms have been developed for fitting AR models to data. To evaluate the performance of AR techniques in different applications it is often necessary to evaluate the accuracy of the AR parameter estimates obtained from a given amount of data. Asymptotic analysis of AR parameter estimation accuracy is a relatively easy task, since the AR model is just a special case of linear regression, except for the initial transient. In particular, the asymptotic Cramer-Rao Bound (CRB) on any unbiased estimate of the AR parameters is well known [1]: it is just the inverse of the covariance matrix of the process, divided by the number of measurements.

For short data records, the actual CRB differs from the asymptotic expression, due to the initial transient of the linear regression. In this note we derive an explicit expression for the Fisher information matrix associated with a finite number of measurements of a Gaussian AR process. The information matrix is shown to be the sum of a constant matrix and the matrix appearing in the asymptotic approximation. The CRB is then given as the inverse of the information matrix. It is shown that the exact CRB can be either larger or smaller than its asymptotic approximation, depending on the process bandwidth. In particular, narrowband processes tend to have CRB's which are considerably smaller than the respective asymptotic approximations.

In the next section we derive the formulas for the exact CRB, and in section 3 we provide some numerical examples illustrating the behavior of this bound.

It should be noted that the CRB is not necessarily a tight bound for a finite numer of data points. Thus, the best achievable parameter estimation accuracy may be considerably poorer than predicted by the CRB when the number of data points is small.

2. DERIVATION OF THE EXACT CRB

Let $\{y_t\}$ be a n-th order stationary Gaussian AR process, defined via the difference equation

$$y_t = -\sum_{k=1}^{n} a_k y_{t-k} + u_t \tag{1}$$

where $\{u_t\}$ is a stationary zero-mean Gaussian white noise with variance σ^2. Let us denote the process covariances by

$$r_k = r_{-k} = E\{y_t \; y_{t-k}\}, \quad -\infty < k < \infty . \tag{2}$$

The covariances are known to satisfy the so-called Yule-Walker equations

$$r_\ell + \sum_{k=1}^{n} a_k r_{\ell-k} = \begin{cases} \sigma^2 & ; \quad \ell = 0 \\ 0 & ; \quad \ell > 0 \end{cases} . \tag{3}$$

Let R_N denote the $N \times N$ symmetric Toeplitz matrix (where $N > n$)

$$(R_N)_{i,j} = r_{i-j} \quad , \quad 1 \le i,j \le N . \tag{4}$$

Also denote

$$S_N = R_N^{-1} . \tag{5}$$

The following three lemmas will be needed to derive the main result.

<u>Lemma 1:</u> The matrix $(\sigma^2)^{-1}R_N$ admits the Cholesky (lower/upper) factorization

$$(\sigma^2)^{-1}R_N = L_N^{-1} L_N^{-T} , \tag{6}$$

where

$$
L_N = \begin{array}{c} \begin{array}{cc} \overbrace{\hphantom{a_n \cdots a_1}}^{n} & \overbrace{\hphantom{1 \cdots \cdots 0}}^{N-n} \end{array} \\ \left[\begin{array}{ccc|ccc} & L_n & & & 0 & \\ \hline a_n & \cdots & a_1 & 1 & & 0 \\ & a_n & \cdots & a_1 & 1 & \\ 0 & & a_n & \cdots & a_1 & 1 \end{array}\right] \end{array} \begin{array}{l} \left.\vphantom{L_n}\right\} n \\ \left.\vphantom{\begin{array}{c} a \\ a \\ a \end{array}}\right\} N-n \end{array} \tag{7}
$$

and L_n^{-1} is the lower Cholesky factor of R_n .

The lemma is proven by computing $(\sigma^2)^{-1} L_N R_N$ and using eq. (3). The result turns out to be an upper triangular matrix with 1's along the diagonal elements starting at the (n+1,n+1) position. Equation (6) then follows from the uniqueness of the Chloesky decomposition.

Lemma 2: The inverse covariance matrix $S_n (= R_n^{-1})$ is given by the expression

$$
S_n = (\sigma^2)^{-1}(A_1 A_1^T - A_2 A_2^T) , \tag{8}
$$

where A_1 and A_2 are the lower triangular Toeplitz matrices

$$
(A_1)_{i,j} = \begin{cases} 1 & ; \quad i=j \\ a_{i-j} & ; \quad i > j \\ 0 & ; \quad i < j \end{cases} ; \quad (A_2)_{i,j} = \begin{cases} a_{n-i+j} & ; \quad i > j \\ 0 & ; \quad i < j \end{cases} . \tag{9}
$$

This is the so-called Gohberg-Semencul formula, proven e.g., in [2].

Lemma 3: Let Y be a zero-mean Gaussian vector whose covariance matrix Λ depends on a vector of parameters θ of dimension m. Then the mxm Fisher information matrix of Y is given by

$$
(J)_{k,\ell} = \frac{1}{2} \operatorname{tr}\left\{\frac{\partial \Lambda^{-1}(\theta)}{\partial \theta_k} \Lambda(\theta) \frac{\partial \Lambda^{-1}(\theta)}{\partial \theta_\ell} \Lambda(\theta)\right\} , \tag{10}
$$

where $\operatorname{tr}\{\cdot\}$ denotes the trace operator.

This formula can be obtained by direct computation, or see e.g., [3].

We now state and prove the main result, as follows:

Theorem 1: The Fisher information matrix corresponding to N consecutive measurements of the given AR process (where $N > n$), is given by the exact expession

$$J_N = \bar{J} + (N-n) \begin{array}{c} \left[\begin{array}{c|c} (2\sigma^4)^{-1} & 0 \\ \hline 0 & (\sigma^2)^{-1}R_n \end{array}\right] \begin{array}{l} \}1 \\ \}n \end{array} \\ \underbrace{\qquad}_{1} \quad \underbrace{\qquad\qquad}_{n} \end{array} \tag{11}$$

where $\bar{J}$ is a constant matrix whose elements are given by

$$(\bar{J})_{1,1} = (2\sigma^4)^{-1}n \tag{12a}$$

$$(\bar{J})_{1,k+1} = (\bar{J})_{k+1,1} = -(2\sigma^2)^{-1}\mathrm{tr}\{\frac{\partial S_n}{\partial a_k} R_n\} \tag{12b}$$

$$(\bar{J})_{k+1,\ell+1} = \frac{1}{2}\,\mathrm{tr}\{\frac{\partial S_n}{\partial a_k} R_n \frac{\partial S_n}{\partial a_\ell} R_n\} . \tag{12c}$$

Proof: By (5),(6) we have

$$S_N = (\sigma^2)^{-1} L_N^T L_N . \tag{13}$$

Hence

$$\frac{\partial S_N}{\partial \sigma^2} = -(\sigma^4)^{-1} L_N^T L_N \tag{14}$$

$$\frac{\partial S_N}{\partial a_k} = (\sigma^2)^{-1} \frac{\partial L_N^T}{\partial a_k} L_N + (\sigma^2)^{-1} L_N^T \frac{\partial L_N}{\partial a_k} \tag{15}$$

$$\frac{\partial S_N}{\partial \sigma^2} R_N = -(\sigma^2)^{-1} I_N \tag{16}$$

$$\frac{\partial S_N}{\partial a_k} R_N = \frac{\partial L_N^T}{\partial a_k} L_N^{-T} + L_N^T \frac{\partial L_N}{\partial a_k} L_N^{-1} L_N^{-T} . \tag{17}$$

After some calculations (using the commutativity of the trace operator and its invariance under transposition) we get:

$$\frac{1}{2}\,\mathrm{tr}\{\frac{\partial S_N}{\partial\sigma^2} R_N \frac{\partial S_N}{\partial\sigma^2} R_N\} = (2\sigma^4)^{-1} N \tag{18}$$

$$\frac{1}{2}\,\mathrm{tr}\{\frac{\partial S_N}{\partial\sigma^2} R_N \frac{\partial S_N}{\partial a_k} R_N\} = -(\sigma^2)^{-1}\,\mathrm{tr}\{L_N^{-1}\frac{\partial L_N}{\partial a_k}\} \tag{19}$$

$$\frac{1}{2}\,\mathrm{tr}\{\frac{\partial S_N}{\partial a_k} R_N \frac{\partial S_N}{\partial a_\ell} R_N\} =$$

$$= \mathrm{tr}\{L_N^{-1}\frac{\partial L_N}{\partial a_k} L_N^{-1}\frac{\partial L_N}{\partial a_\ell}\} + (\sigma^2)^{-1}\,\mathrm{tr}\{\frac{\partial L_N^T}{\partial a_k}\frac{\partial L_N}{\partial a_\ell} R_N\}\ . \tag{20}$$

Now, the partial derivatives of L_N are given by

$$\frac{\partial L_N}{\partial a_k} = \left[\begin{array}{c:c} \frac{\partial L_n}{\partial a_k} & 0 \\ \hdashline \multicolumn{2}{c}{Z_{n-k}^T} \end{array}\right] \begin{array}{l} \}\ n \\ \}\ N-n \end{array}\ , \tag{21}$$

where Z_m is the down shift matrix

$$(Z_m)_{i,j} = \begin{cases} 1 & ;\ i-j = m \\ 0 & ;\ \text{otherwise} \end{cases}\ . \tag{22}$$

Since both L_N and its partial derivatives are lower triangular, and since the last N-n diagonal entries of $\partial L_N/\partial a_k$ are zero, we get

$$\mathrm{tr}\{L_N^{-1}\frac{\partial L_N}{\partial a_k}\} = \mathrm{tr}\{L_n^{-1}\frac{\partial L_n}{\partial a_k}\} \tag{23}$$

$$\mathrm{tr}\{L_N^{-1}\frac{\partial L_N}{\partial a_k} L_N^{-1}\frac{\partial L_N}{\partial a_\ell}\} = \mathrm{tr}\{L_n^{-1}\frac{\partial L_n}{\partial a_k} L_n^{-1}\frac{\partial L_n}{\partial a_\ell}\}\ . \tag{24}$$

Also,

$$\frac{\partial L_N^T}{\partial a_k}\frac{\partial L_N}{\partial a_\ell}R_N = \left[\begin{array}{c|c} \frac{\partial L_n^T}{\partial a_k} & \\ \hline 0 & Z_{n-k} \end{array}\right]\left[\begin{array}{c|c} \frac{\partial L_n}{\partial a_\ell} & 0 \\ \hline \multicolumn{2}{c}{Z_{n-\ell}^T} \end{array}\right]\left[\begin{array}{c|c} R_n & R_{12} \\ \hline R_{21} & R_{N-n} \end{array}\right]$$

$$= \left[\begin{array}{c|c} \frac{\partial L_n^T}{\partial a_k}\frac{\partial L_n}{\partial a_\ell}R_n & \frac{\partial L_n^T}{\partial a_k}\frac{\partial L_n}{\partial a_\ell}R_{12} \\ \hline 0 & 0 \end{array}\right] + Z_{n-k}\,Z_{n-\ell}^T\,R_N \,. \tag{25}$$

Hence

$$tr\{\frac{\partial L_N^T}{\partial a_k}\frac{\partial L_N}{\partial a_\ell}R_N\} = tr\{\frac{\partial L_n^T}{\partial a_k}\frac{\partial L_n}{\partial a_\ell}R_n\} + (N-n)r_{k-\ell} \tag{26}$$

Substituting (24), (25) in (20) yields

$$\frac{1}{2}\,tr\{\frac{\partial S_N}{\partial a_k}R_N\frac{\partial S_N}{\partial a_\ell}R_N\} =$$

$$= tr\{L_n^{-1}\frac{\partial L_n}{\partial a_k}L_n^{-1}\frac{\partial L_n}{\partial a_\ell}\} + (\sigma^2)^{-1}\,tr\{\frac{\partial L_n^T}{\partial a_k}\frac{\partial L_n}{\partial a_\ell}R_n\} + (\sigma^2)^{-1}(N-n)r_{k-\ell}\,. \tag{27}$$

Finally we see that (10) and (18) yield (12a); (10), (19) and (23) yield (12b); and (10), (20) and (27) yield (12c).

Corollary: The exact Cramer-Rao bound on any unbiased estimate of the AR parameters is given by J_N^{-1}, where J_N is given by (11), (12).

Comments:

(1) Note that formulas (11), (12) are actually closed-form expressions for J_N. R_n is given by (cf. (8))

$$R_n = \sigma^2(A_1A_1^T - A_2A_2^T)^{-1} , \tag{28}$$

while $\frac{\partial S_n}{\partial a_n}$ is given by

$$\frac{\partial S_n}{\partial a_n} = (\sigma^2)^{-1}(Z_kA_1^T + A_1Z_k^T - Z_{n-k}A_2^T - A_2Z_{n-k}^T) . \tag{29}$$

(2) When $N \to \infty$, the constant matrix $\bar{J}$ in (11) becomes negligible with respect to the second term. Hence we get the well known asymptotic result [1]

$$\lim_{N\to\infty} N^{-1}J_N = \left[\begin{array}{c|c} (2\sigma^4)^{-1} & 0 \\ \hline 0 & (\sigma^2)^{-1}R_n \end{array}\right] . \tag{30}$$

(3) The difference between the exact information J_N and its asymptotic approximation (30) is not necessarily either positive or negative definite, but can be indefinite in general. Therefore the exact CRB of the AR parameters for short data records can be either larger or smaller than its corresponding asymptotic approximation. Some examples are given in the next section.

3. SOME EXAMPLES

Let us first consider the case of first order AR process with parameters $\{\sigma^2, a\}$. In this case we get*

$$J_N = \begin{bmatrix} \frac{N}{2\sigma^4} & \frac{a}{\sigma^2(1-a^2)} \\ \frac{a}{\sigma^2(1-a^2)} & \frac{2a^2}{(1-a^2)^2} + \frac{N-1}{1-a^2} \end{bmatrix} . \tag{31}$$

It is of interest to examine the ratio of the exact CRB's on σ^2 and a to their respective asymptotic approximations. We denote by $b_N(\theta_k)$ the exact CRB of the parameter θ_k , and by $\tilde{b}_N(\theta_k)$ the asymptotic approximation of the bound. Inverting the matrix in (31) and using the diagonal entries of the inverse we get

$$\frac{b_N(\sigma^2)}{\tilde{b}_N(\sigma^2)} = \frac{N(N-1)(1-a^2) + 2Na^2}{N(N-1)(1-a^2) + 2(N-1)a^2} \tag{32}$$

$$\frac{b_N(a)}{\tilde{b}_N(a)} = \frac{N^2(1-a^2)}{N(N-1)(1-a^2) + 2(N-1)a^2} \tag{33}$$

Eq. (32) clearly implies that the ratio of the bounds on σ^2 is always greater than 1, i.e., the exact bound approaches the asymptotic bound from above. From eq. (33) we get

$$\frac{b_N(a)}{\tilde{b}_N(a)} < 1 \text{ if and only if } |a| > \sqrt{\frac{N}{3N-2}} . \tag{34}$$

Thus we distinguish among three different cases:

(i) $|a| > (2)^{-1/2}$; in this case the exact bound on a is always smaller than the asymptotic bound.

(ii) $|a| < (3)^{-1/2}$; in this case the exact bound is always greater than the asymptotic bound.

*This formula was also given in [4].

(iii) $(3)^{-1/2} < |a| < (2)^{-1/2}$; in this case the exact bound is greater than the asymptotic bound for small values of N, and then changes direction and becomes smaller than the asymptotic bound for large values of N.

A similar behavior is observed for second order AR processes. The explicit formulas are too complicated to analyze by inspection, so one has to resort to numerical evaluation of the CRB's. We tested several second order processes with complex poles of varying magnitudes and a constant phase angle of 45°. Figures 1, 2 and 3 show the results for $\{\sigma^2, a_1, a_2\}$, for the three test cases specified in Table 1.

Table 1: Three Test Cases of Second Order Processes

Test Case	σ^2	a_1	a_2
1	1	-1.378	0.95
2	1	-1.183	0.7
3	1	-0.447	0.1

Case no. 1 corresponds to a narrowband process, while case no. 3 - to a broadband process. As we see, the exact bound on σ^2 is always greater than the asymptotic approximation. The behavior of the bounds on the AR parameters a_1 and a_2 depends on the nature of the process. It appears that for narrowband processes the exact CRB's approach the asymptotic approximations from below, while the opposite is true for broadband processes. In case no. 2, which represents an intermediate bandwidth process, the behavior of the bound changes direction as N increases.

4. CONCLUSIONS

An explicit formula was derived for the Cramer-Rao bound on unbiased estimates of the parameters of Gaussian AR processes. The formula contains a term linear in the number of measurements, plus a constant term. The additional constant term is indefinite in general, so the exact CRB can be either larger or smaller than the corresponding asymptotic approximation.

A common problem in random signal processing is that of estimating narrowband signals from short data records. We have demonstrated that in such situations, the actual CRB can be much smaller than the asymptotic approximation. It is therefore recommended that in analyzing AR algorithms for short data records, comparison should be made to the bound derived here, rather than to the more commonly used asymptotic bound.

We finally note that the result derived in this note apparently does not carry over to moving-average and ARMA processes. These two models are not linear regressions, hence the information matrix is not likely to depend linearly on the number of data points. Formula (10) can still be used to compute J for any desired value of N. However, the amount of computations is proportional to N^3, so this may not be convenient in practice. Asymptotic CRB formulas for the ARMA and AR-plus-Noise cases can be found in [5].

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Figure 1: The Ratio of the Bounds on σ^2.

Figure 2: The Ratio of the Bounds on a_1.

Figure 3: The Ratio of the Bounds on a_2.

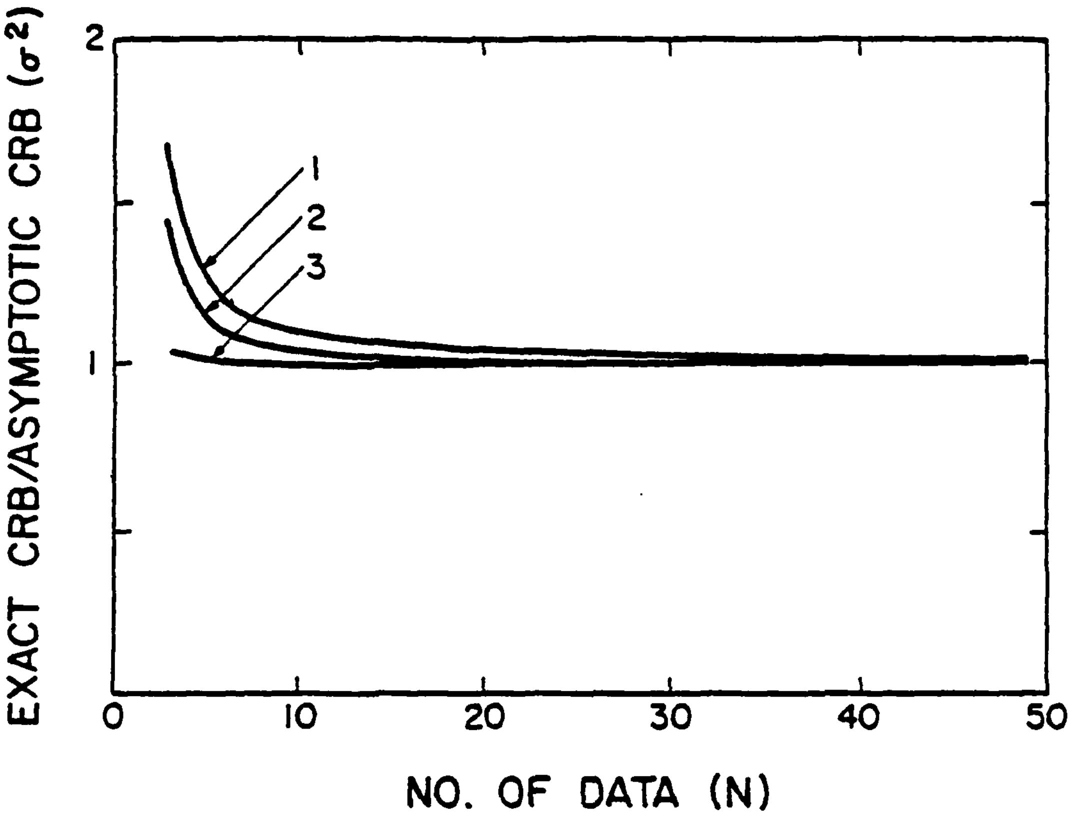
EXACT CRB/ASYMPTOTIC CRB (σ^2)
2
1
0
10
20
30
40
50
NO. OF DATA (N)
1
2
3

FIGURE 1

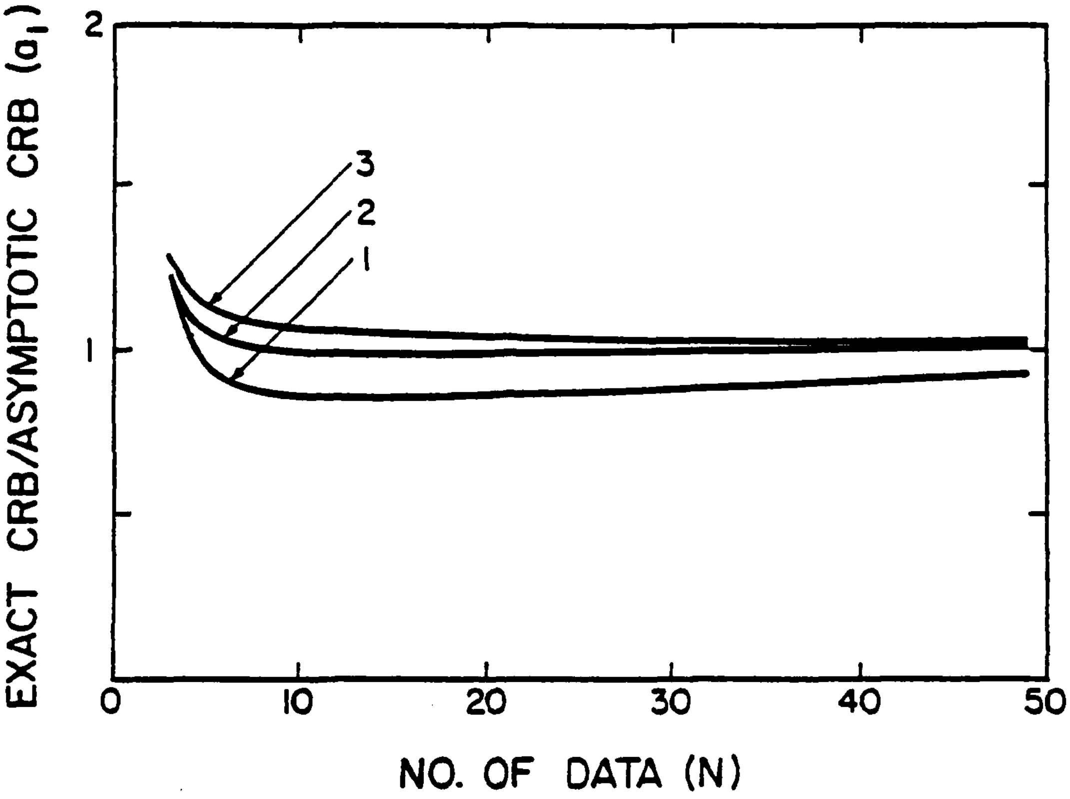

FIGURE 2

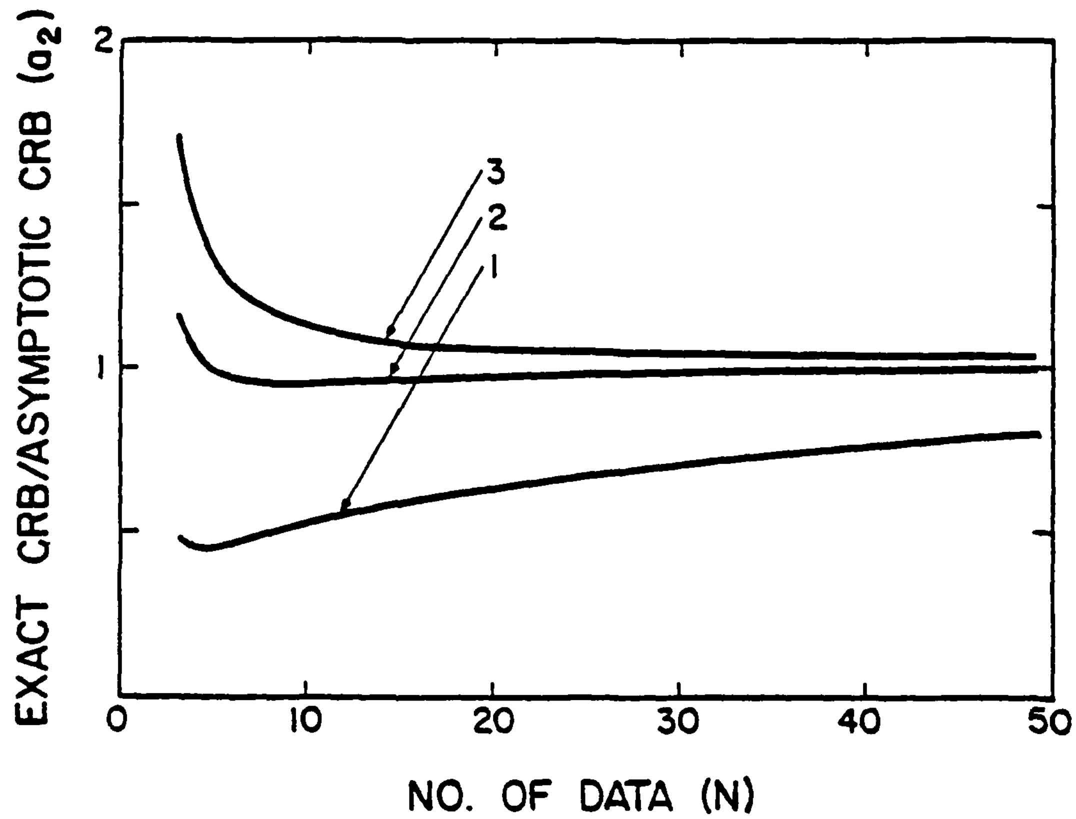

FIGURE 3

APPENDIX L

ON INSTRUMENTAL VARIABLE ESTIMATION OF SINUSOID FREQUENCIES AND THE PARSIMONY PRINCIPLE

Revision 10/85

ON INSTRUMENTAL VARIABLE ESTIMATION OF SINUSOID FREQUENCIES AND THE PARSIMONY PRINCIPLE

Petre Stoica, Benjamin Friedlander, and Torsten Söderström

ABSTRACT

Mulitple sinusoids in noise can be modeled as an ARMA process with the AR parameters satisfying certain symmetry relations. According to the "parsimony principle" the constraints on the AR parameters should be taken into account to get improved estimation accuracy. It is shown in this note that when estimating the AR parameters by a general instrumental-variable method, such a parsimony does not necessarily apply. However, the parsimony principle does hold when an optimal instrumental-variable method is used.

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1. INTRODUCTION

A sinusoids in noise process obeys an ARMA equation of a special structure [1,2,11]. In particular, the AR parameters of this ARMA equation possess a certain symmetry property [1,5,11]. Taking this symmetry into account should presumably result in improved estimation accuracy. Since the AR parameters contain complete information on the sinusoid frequencies, their accurate estimation is important.

A frequently used technique for estimating the AR parameters of an ARMA is the MYW method [1,2,3] which is closely related to the more general class of instrumental variable (IV) methods [14]. Within this technique, the symmetry which the AR parameters must satisfy may be ignored [2,3] or taken into account [1,5,11]. The computational burdens that result in either case are comparable [5]. However, we may expect that better accuracy should be obtained in the second case. This would presumably follow from the so-called "parsimony principle" [8,9].

Our aim here is to investigate this conjecture. Taking into account the symmetry of the AR parameters may lead often to improved estimates. However, we show by means of a counter-example that this is not always true, contrary to what is sometimes stated in the literature [11]. We also show that if an optimal instrumental variable method is used (closely related to the MYW method with an optimal weighting matrix) then the parsimony principle does hold.

2. MAIN RESULTS

Consider the following sinusoidal signal

$$x(t) = \sum_{k=1}^{m} \alpha_k \sin(\omega_k t + \phi_k) \tag{2.1}$$

where

$$\alpha_k \, , \, \phi_k \in R, \; \omega_k \in (0, \pi),$$

$$\omega_i \neq \omega_j \quad \text{for} \quad i \neq j$$

Let y(t) denote the noise-corrupted measurements of x(t).

$$y(t) = x(t) + e(t), \qquad t = 1,2,..., \tag{2.2}$$

where e(t) is a sequence of independent and identically distributed random variables with zero mean and variance λ^2. We assume that x(t) and e(s) are uncorrelated for any t and s.

As is well known, x(t) obeys the following autoregressive equation [1, 2, 11],

$$x(t) + a_1 x(t-1) + ... + a_n x(t-n) = 0, \qquad n \triangleq 2m \tag{2.3}$$

where $\{a_i\}$ are defined by

$$1 + a_1 z + ... + a_n z^n = \prod_{k=1}^{m} (1 - 2 \cos \omega_k \, z + z^2) \triangleq A(z) \; . \tag{2.4}$$

Since A(z) has complex-conjugate unit-modulus roots, we must have

$$a_i = a_{n-i}, \quad i = 0, ..., n, \quad (a_0 \triangleq 1) \; . \tag{2.5}$$

It follows from (2.2) and (2.3) that

$$A(q^{-1})\, y(t) = A(q^{-1}) e(t) \ , \tag{2.6}$$

which can be written as

$$y(t) = \phi^T(t)\theta + A(q^{-1})\, e(t) \ , \tag{2.7}$$

where

$$\phi(t) = -[y(t-1)\ \dots\ y(t-n)]^T, \tag{2.8}$$

$$\theta = [a_1\ \dots\ a_n]^T \ .$$

If the constraint (2.5) is taken into account, then (2.7) becomes

$$y(t) + y(t-n) = \psi^T(t)\ \alpha + A(q^{-1})\ e(t), \tag{2.9}$$

where

$$\psi(t) = -[\{y(t-1) + y(t-n+1)\},\ \dots,\ \{y(t-m+1) + y(t-m-1)\},\ \{y(t-m)\}]^T$$

$$\alpha = [a_1\ \dots\ a_m]^T \ . \tag{2.10}$$

Consider the following instrumental variable (IV) estimates of θ :

(i) Unconstrained IV estimate

$$\hat{\theta} = \arg\min_{\theta} \| [\frac{1}{N} \sum_{t=1}^{N} z(t)\phi^T(t)]\theta - [\frac{1}{N} \sum_{t=1}^{N} z(t)y(t)] \|_Q^2 \ , \ Q > 0 \tag{2.11a}$$

where N denotes the number of data points and $z(t)$ is the IV vector given by

$$z(t) = [y(t-n-1) \ \dots \ y(t-n-M)]^T, \qquad M > n \ . \tag{2.11b}$$

or equivalently, $\hat{\theta}$ is the least squares solution of

$$Q^{1/2}\left[\frac{1}{N}\sum_{t=1}^{N} z(t)\varphi^T(t)\right]\theta = Q^{1/2}\left[\frac{1}{N}\sum_{t=1}^{N} z(t)y(t)\right] \tag{2.12}$$

(ii) Constrained IV estimate

Note from (2.5) that

$$\theta = U\alpha + e_{2m} \tag{2.13a}$$

where

$$e_k = [0, \dots, 0, 1]^T = \text{unit vector of length } k \tag{2.13b}$$

and

$$U = \begin{bmatrix} 1 & & & \\ & \ddots & 0 & \\ & & 1 & \\ 0 & & 0 & 1 \\ & & 1 & 0 \\ & \cdot^{\cdot^{\cdot}} & 0 & \vdots \\ 1 & & & \\ 0 & \dots & \dots & 0 \end{bmatrix}, \qquad (2m \times m) \tag{2.13c}$$

The constrained estimate $\hat{\hat{\theta}}$ is defined by (2.11), under the constraint (2.13). Thus,

$$\hat{\alpha} = \arg\min_{\alpha} \left\| \left[\frac{1}{N}\sum_{t=1}^{N} z(t)\varphi^T(t)\right]\alpha - \left[\frac{1}{N}\sum_{t=1}^{N} z(t)\{y(t)+y(t-n)\}\right] \right\|_Q^2$$

$$\psi^T(t) = \phi^T(t)U$$

$$(2.14)$$

$$\hat{\hat{\theta}} = U\,\hat{\alpha} + e_{2m}$$

or equivalently, $\hat{\alpha}$ is the least-squares solution of

$$Q^{1/2}\left[\frac{1}{N}\sum_{t=1}^{N} z(t)\psi^T(t)\right]\alpha = Q^{1/2}\left[\frac{1}{N}\sum_{t=1}^{N} z(t)\{y(t)+y(t-n)\}\right] \tag{2.15}$$

Note that by the transformation in (2.13), we have converted the constrained optimization problem for θ into an unconstrained optimization problem for α (2.14). We could also have obtained $\hat{\hat{\theta}}$ by using the results of the theory of least-squares regression with linear constraints [13]. However, the formula provided by this theory for $\hat{\hat{\theta}}$, though equivalent to (2.14), is more complicated [12,13].

The IV estimates (2.11) and (2.14) are asymptotically equivalent to some MYW estimators which are easier to implement. There are various interesting computational issues related to (2.11) and (2.14), or rather to the asymptotically equivalent MYW estimators, for which we refer to [1-5, 10, 11].

We are interested in comparing the accuracies of the two estimates $\hat{\theta}$ and $\hat{\hat{\theta}}$. To do this we rely on the following asymptotic results which follow from the general theory developed in [6][7][12]. The asymptotic distributions of the normalized IV estimation errors are given by:

$$\frac{\sqrt{N}}{\lambda}(\hat{\theta}-\theta) \xrightarrow[N \to \infty]{\text{distribution}} \mathcal{N}(0, P_{\hat{\theta}}) \tag{2.16}$$

where

$$P_{\hat{\theta}} = (R^T QR)^{-1} R^T QSQR(R^T QR)^{-1}, \tag{2.17}$$

$$R = E\{z(t)\phi^T(t)\} , \tag{2.18}$$

$$S = E\{A(q^{-1})z(t)\ A(q^{-1})z^T(t)\} =$$

$$= E\{A(q^{-1}) \begin{bmatrix} e(t-1) \\ \vdots \\ e(t-M) \end{bmatrix} \cdot A(q^{-1})[e(t-1)\ \ldots\ e(t-M)]\} \tag{2.19}$$

and

$$\frac{\sqrt{N}}{\lambda}(\hat{\alpha} - \alpha) \xrightarrow[N \to \infty]{\text{distribution}} \mathcal{N}(0,\ P_{\hat{\alpha}}) \tag{2.20}$$

where

$$P_{\hat{\alpha}} = (\tilde{R}^T Q\tilde{R})^{-1}\ \tilde{R}^T QSQ\tilde{R}(\tilde{R}^T Q\tilde{R})^{-1} , \tag{2.21}$$

$$\tilde{R} = E\{z(t)\ \psi^T(t)\} = RU . \tag{2.22}$$

The last result implies that

$$\frac{\sqrt{N}}{\lambda}\ (\hat{\hat{\theta}} - \theta) \xrightarrow[N \to \infty]{\text{distribution}} \mathcal{N}(0,\ P_{\hat{\hat{\theta}}}) \tag{2.23}$$

where

$$P_{\hat{\hat{\theta}}} = U\ P_{\hat{\alpha}}\ U^T = U(U^T R^T QRU)^{-1} U^T R^T Q\ SQRU(U^T R^T QRU)^{-1} U^T \tag{2.24}$$

The covariance matrices $P_{\hat{\theta}}$ and $\theta_{\hat{\hat{\theta}}}$ depend on Q. It can be shown [12], [15] that $P_{\hat{\theta}}$ and $P_{\hat{\hat{\theta}}}$ are bounded from below by

$$\tilde{P}_{\hat{\theta}} = (R^T S^{-1} R)^{-1} \tag{2.25}$$

and

$$\tilde{P}_{\hat{\hat{\theta}}} = U(U^T R^T S^{-1} R\ U)^{-1} U^T \tag{2.26}$$

Furthermore, it is straightforward to show that the lower bounds above are attained for

$$Q = S^{-1} \tag{2.27}$$

The IV estimates (2.11), (2.14) with the optimal weighting matrix above are called optimal IV estimators. For a discussion of their implementation see [12],[16].

The above covariance matrices are useful in evaluating the performances of the two estimates (2.11) and (2.14) in specific cases. For the case of "standard" ARMA processes, an extensive analytical study of performance of the MYW estimators has been reported recently in [4]. For the sinusoids in noise process, no similar study of performance of estimates like (2.11) and (2.14) appears to be available in literature.

The question we want to address here is whether $P_{\hat{\theta}} > P_{\hat{\hat{\theta}}}$. First we note that the problem under study is related to the theory of least-squares regression with linear constraints. For $Q = I$, $P_{\hat{\theta}}$ and $P_{\hat{\hat{\theta}}}$ can be interpreted as the covariance matrices of the constrained and unconstrained least-squares

estimates of the parameters of a regression model with R being the "regressor matrix" and S being the covariance matrix of the residuals. It is then known that $P_{\hat{\theta}} > P_{\hat{\hat{\theta}}}$ if S=I [13]. It was conjectured in [11] that $P_{\hat{\theta}} > P_{\hat{\hat{\theta}}}$ also for the case under consideration where $S \neq I$ (and Q=I). However, no formal analysis of the case $S \neq I$ seems to be available in the literature.

In the following we show by means of a counterexample that for $S \neq I$ and $Q = I$ the inequality $P_{\hat{\theta}} > P_{\hat{\hat{\theta}}}$ does not necessarily hold. Note that $P_{\hat{\theta}} > P_{\hat{\hat{\theta}}}$ implies that

$$\overline{P}_{\hat{\alpha}} = [I_m, 0] P_{\hat{\theta}} \begin{bmatrix} I_m \\ 0 \end{bmatrix} > [I_m, 0] P_{\hat{\hat{\theta}}} \begin{bmatrix} I_m \\ 0 \end{bmatrix} = P_{\hat{\alpha}} \tag{2.28}$$

The next example shows that $\overline{P}_{\hat{\alpha}} > P_{\hat{\alpha}}$ does not always hold, thus contradicting the inequality $P_{\hat{\theta}} > P_{\hat{\hat{\theta}}}$.

<u>Example: A Single Sinusoid-in-Noise Process</u>

We evaluated the covariances $P_{\hat{\alpha}}$ and $P_{\hat{\alpha}}$ for $Q = I$,

$m = 1$; $\alpha_1 = \sqrt{2}$; $\phi_1 = 0$; $\lambda^2 = 1$; $M = 2$, and $\omega_1 \in [0.12\pi, 0.88\pi]$

To evaluate $P_{\hat{\theta}}$ or $P_{\hat{\alpha}}$ we need to compute the covariances of x(t). These are given by the well-known formula [10,11]

$$E\{x(t)\, x(t+k)\} = \sum_{j=1}^{m} \frac{\alpha_j^2}{2} \cos k\omega_j \, . \tag{2.29}$$

In Figure 1, we plot $\log \overline{P}_{\hat{\alpha}}$ and $\log P_{\hat{\alpha}}$ versus ω. For $\omega < 0.2\pi$ and $\omega > 0.8\pi$, the IV estimate (2.14) has much better accuracy than (2.11). The poor accuracy of (2.11) for such values of ω was expected since the matrix $R^T R$ in (2.17) is nearly singular for ω close to 0 or π.

For ω in the range $[0.2\pi, 0.8\pi]$, the accuracies of the two estimates

are comparable. Moreover, for ω in the intervals shown in Figure 1, the estimate (2.11) is more accurate than (2.14), which concludes the counter-example.

Next we show that the parsimony principle does apply to the case of optimal IV estimates, in the sense that

$$\bar{P}_{\hat{\theta}} > \bar{P}_{\hat{\hat{\theta}}} \tag{2.30}$$

This follows from the theory of least-squares regression with linear constraints for the case of uncorrelated residuals [13]. The materices $\bar{P}_{\hat{\theta}}$ and $\bar{P}_{\hat{\hat{\theta}}}$ can be interpreted as the covariance matrices of the constrained and unconstrained least-squares estimates of the parameters of a regression with $S^{-1/2}R$ as the regressor matrix and with uncorrelated residuals. For the case considered here we have a simple proof of (2.30) which we include for completeness.

Proof of (2.30):

$$\bar{P}_{\hat{\theta}} > \bar{P}_{\hat{\hat{\theta}}} \Longleftrightarrow R^TS^{-1}R - (R^TS^{-1}R)U(U^TR^TS^{-1}RU)^{-1}U^T(R^TS^{-1}R) > 0$$

$$\Longleftrightarrow \begin{bmatrix} R^TS^{-1}R & (R^TS^{-1}R)U \\ U^T(R^TS^{-1}R) & U^T(R^TS^{-1}R)U \end{bmatrix} = \begin{bmatrix} I \\ U^T \end{bmatrix} (R^TS^{-1}R)[I \;\; U] > 0$$

3. CONCLUSIONS

It was shown that the parsimny principle does not hold in general when an IV (or MYW) method is used to estimate the parameters of sinusoid-in-noise type models. However, when an optimal IV (or an optimal MYW) method is used, the parsimony principle does hold. This result is interesting from a theoretical standpoint and helps to clarify some conjectures made in the

literature.

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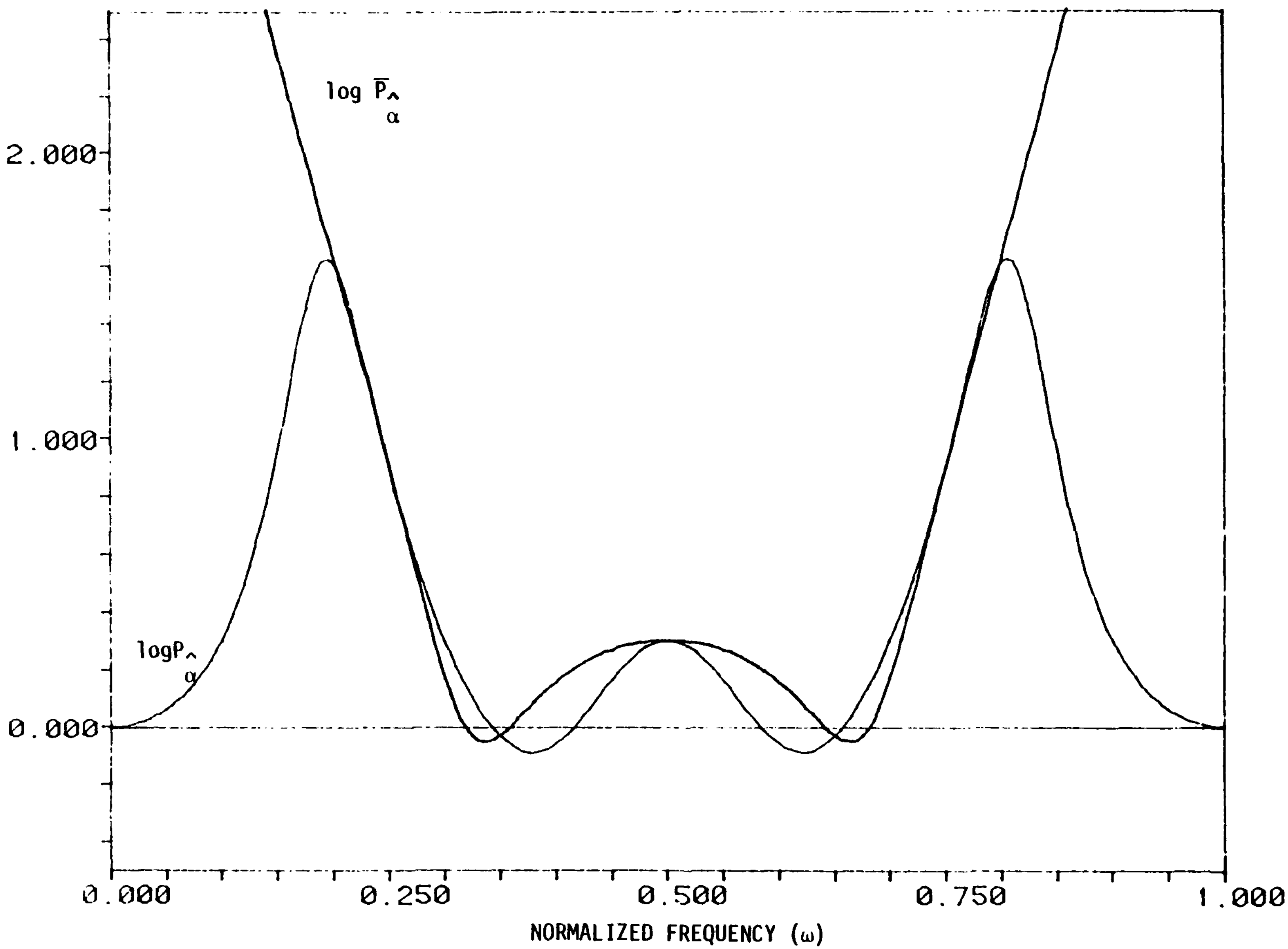

Figure 1: Comparison of the Accuracies of the Constrained and Unconstrained IV Estimates

APPENDIX M

PARAMETER ESTIMATION OF CONTINUOUS-TIME STATIONARY GAUSSIAN PROCESSES WITH RATIONAL SPECTRA

PARAMETER ESTIMATION OF CONTINUOUS-TIME STATIONARY
GAUSSIAN PROCESSES WITH RATIONAL SPECTRA

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ABSTRACT

This paper considers the problem of estimating the parameters of continuous-time stationary Gaussian processes with rational spectra, from uniformly sampled measurements. The sampled process is shown to be an autoregressive moving-average process, and explicit relationships between the parameters of the continuous-time and the sampled processes are derived. These relationships are then used to derive a lower bound on unbiased estimates of the continuous-time parameters, and on the generalized variance of such estimates. It is shown by some examples that the bound on the generalized variance depends on the sampling interval in a non-monotonic manner. In particular, for each specific set of parameters, there exists a sampling interval for which the lower bound is minimized.

This work was supported by the Army Research Office under contract number DAAG29-83-C-0027.

1. INTRODUCTION

Digital processing of continuous-time signals involves the sampling of these signals. Most often the sampling is uniform, i.e., the sampling interval is constant. In some cases the user is interested in modeling the sampled signal, rather than the original continuous-time signal, while in other cases a model of the continuous-time signal is required. A typical situation is that of a signal generated by a physical system whose mathematical model is known, but whose parameters are unknown (such as a mechanical system with unknown masses, viscosities and spring coefficients). In such cases the primary goal of the digital processing is to identify the parameters of the original continuous-time system.

In this paper we consider a special class of continuous-time signals: stationary Gaussian processes with rational spectra. It is well known that uniform sampling of such processes gives rise to autoregressive moving-average (ARMA) processes of order equal to the denominator degree of the spectral density of the continuous-time process. The achievable accuracy in estimating the parameters and the power spectral densities of ARMA processes was studied in [1],[2]. In this paper we give quantitative answers to the following questions: i) What is the achievable accuracy in estimating the parameters of the continuous-time spectral density from the sampled ARMA process? ii) How is the achievable accuracy affected by the choice of sampling rate?

We assume that the number of data points of the sampled signal is fixed, i.e., that the total interval over which data are collected is proportional to the sampling interval. This is a reasonable assumption since it is often desired to process data in batches of a fixed size. We also assume that the parameter estimation method used is unbiased, at least asymptotically (e.g., the maximum likelihood estimator). Under these assumptions, we show that the lower bound on the generalized variance of the continuous-time parameter estimates is a non-monotonic function of the sampling interval. Consequently, for any given set of parameters there exists a sampling interval for which the generalized variance is minimal. The range of sampling rates for which the generalized variance is nearly minimal (the flat region of the curve) can be small or large, depending on the characteritics of the given signal.

The outline of the paper is as follows. In section 2 we derive closed-form expressions for the parameters of the sampled process as a function of the parameters of the given process. In section 3 we derive a lower bound on the variances of unbiased estimates of the continuous-time parameters. In section 4 we illustrate the existence of an optimal sampling rate, and examine the dependence of the generalized variance on the sampling interval for some examples. In section 5 we discuss potential applications of the results of this paper. The reader may want to skip sections 2 and 3 and go directly to section 4 on the first reading.

2. DISCRETIZATION OF THE CONTINUOUS-TIME SPECTRUM

Let $x(t)$ be a continuous-time Gaussian stationary random process. The process is assumed to have zero mean and a rational power spectral density function

$$S_x(s) = \frac{\beta(s)\,\beta(-s)}{\alpha(s)\alpha(-s)} \quad , \tag{1}$$

where

$$a(s) = s^n + \alpha_1 s^{n-1} + \dots + \alpha_n \ ; \quad \beta(s) = \beta_1 s^{n-1} + \dots + \beta_n \ .$$

Both polynomials are assumed to have all their roots in the left half plane. Also, to simplify the analysis, we restrict ourselves to the case where all the roots of $\alpha(s)$ are distinct. Note that the degree of $\beta(s)$ is strictly less than that of $\alpha(s)$. This means that $x(t)$ does not contain a white noise component.

Assume that $x(t)$ is sampled at multiples of the sampling interval T, to yield a discrete-time Gaussian stationary process $\{y_k\}$, where

$$y_k = x(kT) \ , \ k = \dots -1,0,1,\dots .$$

Our aim is to derive an expression for the power spectral density of $\{y_k\}$, which we will denote by $S_y(z)$. As we will see $\{y_k\}$ turns out to be an autoregressive moving-average (ARMA) process of order (n, n-1).

The continuous-time spectrum, being a symmetric function of s, can be decomposed as

$$S_x(s) = \frac{\gamma(s)}{\alpha(s)} + \frac{\gamma(-s)}{\alpha(-s)} \quad , \tag{2}$$

where $\gamma(s) = \gamma_1 s^{n-1} + \dots + \gamma_n$. The coefficients of $\gamma(s)$ can be obtained from those of $\alpha(s)$ by equating coefficients in the identity

$$\gamma(s)\alpha(-s) + \gamma(-s)\alpha(s) = \beta(s)\beta(-s) \ . \tag{3}$$

Let us introduce the following notation

$$\underline{\alpha} = [\alpha_1, \alpha_2, \ldots, \alpha_n]^T \ ; \ \underline{\beta} = [\beta_1, \beta_2, \ldots, \beta_n]^T \ ; \ \underline{\gamma} = [\gamma_1, \gamma_2, \ldots, \gamma_n]^T$$

$$\{A\}_{i,j} = \begin{cases} (-1)^{n-j}\alpha_{2i-j} & ; \quad 1 \le 2i-j \le n \\ (-1)^n & ; \quad 2i-j = 0 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

$$\{B\}_{i,j} = \begin{cases} (-1)^{n-j}\beta_{2i-j} & ; \quad 1 \le 2i-j \le n \\ 0 & ; \quad \text{otherwise} \end{cases}$$

where A and B are n x n matrices. Then it can be verified that (3) is equivalent to

$$\underline{\gamma} = \frac{1}{2} A^{-1} B \underline{\beta} \ . \tag{4}$$

Next we express $\gamma(s)/\alpha(s)$ as a partial fraction expansion. Let

$$\alpha(s) = (s-\lambda_1)(s-\lambda_2) \ldots (s-\lambda_n) \ .$$

Since the roots are assumed to be distinct, and since $\gamma(s)/\alpha(s)$ is strictly proper, we have

$$\frac{\gamma(s)}{\alpha(s)} = \sum_{m=1}^{n} \frac{\delta_m}{s-\lambda_m} \ . \tag{5}$$

The coefficients $\{\delta_m, 1 \le m \le n\}$ can be evaluated by multiplying (5) by $(s-\lambda_m)$ and then taking the limit as $s \to \lambda_m$. This yields

$$\delta_m = \frac{\gamma_1 \lambda_m^{n-1} + \ldots + \gamma_n}{\prod_{i \neq m} (\lambda_m - \lambda_i)} \ . \tag{6}$$

We can now use the inverse Laplace transform to get the autocorrelation function of x(t).

$$\sigma_x(\tau) = \sum_{m=1}^{n} \delta_m e^{\lambda_m |\tau|} . \qquad (7)$$

The covariance sequence of the discrete-time process $\{y_k\}$ is obtained by sampling $\sigma_x(\tau)$ at multiples of the interval T,

$$r_y(\ell) \triangleq Ey_k y_{k-\ell} = \sigma_x(\ell T) = \sum_{m=1}^{n} \delta_m e^{\lambda_m T|\ell|} = \sum_{m=1}^{n} \delta_m \mu_m^{|\ell|} , \qquad (8)$$

where

$$\mu_m \triangleq e^{\lambda_m T} , \quad 1 < m < n.$$

The power spectral density of $\{y_k\}$ is defined as the z-transform of the covariance sequence,

$$S_y(z) = \sum_{\ell=-\infty}^{\infty} r_y(\ell) z^{-\ell} = \sum_{\ell=-\infty}^{\infty} \left(\sum_{m=1}^{n} \delta_m \mu_m^{|\ell|} \right) z^{-\ell}$$

$$= \sum_{m=1}^{n} \delta_m \left(\sum_{\ell=-\infty}^{\infty} \mu_m^{|\ell|} z^{-\ell} \right) = \sum_{m=1}^{n} \frac{(1-\mu_m^2)\delta_m}{(1-\mu_m z)(1-\mu_m z^{-1})} . \qquad (9)$$

As we see, $S_y(z)$ is a rational function of z having a reciprocal symmetry. By bringing the terms of the right-hand side of (9) under common denominator, we get

$$S_y(z) = \frac{e(z)}{a(z)a(z^{-1})} , \qquad (10)$$

where

$$a(z) = 1 + a_1 z + \dots a_n z^n = (1-\mu_1 z)(1-\mu_2 z) \dots (1-\mu_n z)$$

$$e(z) = e_{n-1} z^{n-1} + \dots + e_1 z + e_0 + e_1 z^{-1} + \dots + e_{n-1} z^{-(n-1)} .$$

Note that $e(z)$ possesses reciporcal symmetry. Also, from (9) it is clear that $e(z)$ is positive for all $z = e^{j\omega}$, $-\pi < \omega < \pi$. Hence $e(z)$ can be factored as

$$e(z) = \sigma^2 b(z) b(z^{-1}) , \tag{11}$$

where

$$b(z) = 1 + b_1 z + \ldots + b_{n-1} z^{n-1} = (1-\nu_1 z)\ldots(1-\nu_{n-1} z) ,$$

and all $\{\nu_m, 1 < m < n-1\}$ have magnitudes strictly less than one. Finally, the discrete-time power spectral density is given by

$$S_y(z) = \frac{\sigma^2 b(z) b(z^{-1})}{a(z) a(z^{-1})} = \sigma^2 \frac{\sum_{m=1}^{n-1} (1-\nu_m z)(1-\nu_m z^{-1})}{\prod_{m=1}^{n} (1-\mu_m z)(1-\mu_m z^{-1})} . \tag{12}$$

As we see, the discrete-time process $\{y_k\}$ can be modeled as an ARMA process of order $(n,n-1)$,

$$y_k = -\sum_{m=1}^{n} a_m y_{k-m} + u_k + \sum_{m=1}^{n-1} b_m u_{k-m} , \tag{13}$$

where $\{u_k\}$ is the innovation process of $\{y_k\}$ and σ^2 is the variance of $\{u_k\}$.

3. THE BOUND ON THE VARIANCE OF THE ESTIMATES

As we saw in the previous section, the spectral density of the sampled process depends on the parameters $\{\sigma^2, \mu_m, \nu_m\}$, which in turn depend on the parameters $\{\alpha_m, \beta_m\}$ of the continuous-time process. Suppose we have N consecutive measurements of the sampled process, say $\{y_k, 1 < k < N\}$. Since $\{y_k\}$ is an ARMA process, the parameters $\{\sigma^2, a_m, b_m\}$, or equivalently $\{\sigma^2, \mu_m, \nu_m\}$, can be estimated by any of several available techniques (such as maximum likelihood, nonlinear least-squares, pseudo-linear regression). The estimated values of $\{\alpha_m, \beta_m\}$ can then be computed by reversing the procedure described in the previous section. Our aim here is to examine the best possible performance of such a procedure, i.e., to derive a lower bound on the variances of the estimates $\{\alpha_m, \beta_m\}$.

Let us denote by $\theta^{(1)}$ the parameter vector

$$\theta^{(1)} = [\mu_1, \ldots, \mu_n, \nu_1, \ldots, \nu_{n-1}, \sigma^2]^T .$$

The large-sample Fisher information matrix of $\theta^{(1)}$ corresponding to N measurements, is given by [3, p. 242]

$$I_N\{\theta^{(1)}\} =$$

$$N \cdot \left[\begin{array}{ccc|ccc|c}
(1-\mu_1\mu_1^*)^{-1} & \ldots & (1-\mu_1\mu_n^*)^{-1} & (1-\mu_1\nu_1^*) & \ldots & (1-\mu_1\nu_{n-1}^*)^{-1} & 0 \\
\vdots & & \vdots & \vdots & & \vdots & \vdots \\
(1-\mu_n\mu_1^*)^{-1} & \ldots & (1-\mu_n\mu_n^*)^{-1} & (1-\mu_n\nu_1^*) & \ldots & (1-\mu_n\nu_{n-1}^*)^{-1} & 0 \\
\hline
(1-\nu_1\mu_1^*)^{-1} & \ldots & (1-\nu_1\mu_n^*)^{-1} & (1-\nu_1\nu_1^*)^{-1} & \ldots & (1-\nu_1\nu_{n-1}^*)^{-1} & 0 \\
\vdots & & \vdots & \vdots & & \vdots & \vdots \\
(1-\nu_{n-1}\mu_1^*)^{-1} & \ldots & (1-\nu_{n-1}\mu_n^*)^{-1} & (1-\nu_{n-1}\nu_1^*)^{-1} & \ldots & (1-\nu_{n-1}\nu_{n-1}^*) & 0 \\
\hline
0 & \ldots & 0 & 0 & \ldots & 0 & (2\sigma^4)^{-1}
\end{array}\right]$$

The Cramer-Rao lower bound on unbiased estimates of $\theta^{(1)}$ is given by the

inverse of the information matrix,

$$\mathrm{CRB}\{\theta^{(1)}\} = [I_N\{\theta^{(1)}\}]^{-1} \tag{15}$$

We are interested in deriving an expression for the Cramer-Rao bound for the parameter vector

$$\theta^{(6)} = [\alpha_1,\ldots,\alpha_n,\ \beta_1,\ldots,\beta_n]^T .$$

This is related to the CRB for $\theta^{(1)}$ via the formula [4, p. 194]

$$\mathrm{CRB}\{\theta^{(6)}\} = [\frac{\partial\theta^{(6)}}{\partial\theta^{(1)}}]\mathrm{CRB}\{\theta^{(1)}\}[\frac{\partial\theta^{(6)}}{\partial\theta^{(1)}}]^H$$

$$= [\frac{\partial\theta^{(6)}}{\partial\theta^{(1)}}][I_N\{\theta^{(1)}\}]^{-1}[\frac{\partial\theta^{(6)}}{\partial\theta^{(1)}}]^H, \tag{16}$$

where $(\cdot)^H$ denotes Hermitian transpose. Rather than evaluating the Jacobian $\partial\theta^{(6)}/\partial\theta^{(1)}$ directly, it will be convenient to introduce four intermediate vectors, as follows:

$$\theta^{(2)} = [\mu_1,\ \cdots\ ,\ \mu_n,\ \delta_1,\ \cdots\ ,\ \delta_n]^T$$

$$\theta^{(3)} = [\lambda_1,\ \cdots\ ,\ \lambda_n,\ \delta_1,\ \cdots\ ,\ \delta_n]$$

$$\theta^{(4)} = [\lambda_1,\ \cdots\ ,\ \lambda_n,\ \gamma_1,\ \cdots\ ,\ \gamma_n]^T$$

$$\theta^{(5)} = [\alpha_1,\ \cdots\ ,\ \alpha_n,\ \gamma_1,\ \cdots\ ,\ \gamma_n]^T$$

Then we have

$$\frac{\partial\theta^{(6)}}{\partial\theta^{(1)}} = \frac{\partial\theta^{(6)}}{\partial\theta^{(5)}}\frac{\partial\theta^{(5)}}{\partial\theta^{(4)}}\frac{\partial\theta^{(4)}}{\partial\theta^{(3)}}\frac{\partial\theta^{(3)}}{\partial\theta^{(2)}}\frac{\partial\theta^{(2)}}{\partial\theta^{(1)}} . \tag{17}$$

Next we derive explicit expressions for the various partial derivatives in (17). Recall that

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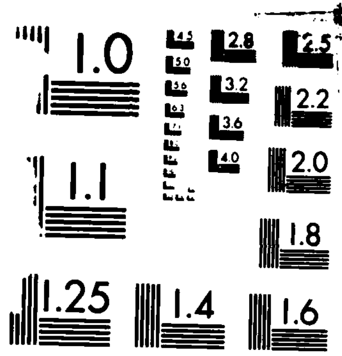

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$$S_y(z) = \sigma^2 \cdot \frac{\prod_{m=1}^{n-1} (1-\nu_m z)(1-\nu_m z^{-1})}{\prod_{m=1}^{n} (1-\mu_m z)(1-\mu_m z^{-1})} = \sum_{m=1}^{n} \frac{(1-\mu_m^2)\delta_m}{(1-\mu_m z)(1-\mu_m z^{-1})} \tag{18}$$

Multiplying both sides by $(1-\mu_k z^{-1})$ and substituting $z = \mu_k$ we get

$$\delta_k = \sigma^2 \cdot \frac{\prod_{m=1}^{n} (1-\nu_m\mu_k)(1-\nu_m\mu_k^{-1})}{\left[\prod_{m=1}^{n} (1-\mu_m\mu_k)\right]\left[\prod_{m\neq k} (1-\mu_m\mu_k^{-1})\right]} , \tag{19}$$

hence

$$\log\delta_k = \log\sigma^2 + \sum_{m=1}^{n} \log(1-\nu_m\mu_k) + \sum_{m=1}^{n} \log(1-\nu_m\mu_k^{-1})$$

$$- \sum_{m=1}^{n} \log(1-\mu_m\mu_k) - \sum_{m\neq k} \log(1-\mu_m\mu_k^{-1}) . \tag{20}$$

Differentiating (20) gives

$$\frac{1}{\delta_k}\frac{\partial\delta_k}{\partial\mu_\ell} = \frac{\mu_k}{1-\mu_\ell\mu_k} + \frac{\mu_k^{-1}}{1-\mu_\ell\mu_k^{-1}} \quad ; \quad \ell \neq k \tag{21a}$$

$$\frac{1}{\delta_k}\frac{\partial\delta_k}{\partial\mu_k} = -\sum_{m=1}^{n} \frac{\nu_m}{1-\nu_m\mu_k} + \sum_{m=1}^{n} \frac{\mu_m\mu_k^{-2}}{1-\nu_m\mu_k^{-1}} + \frac{2\mu_k}{1-\mu_k^2}$$

$$+ \sum_{m\neq k} \frac{\mu_m}{1-\mu_m\mu_k} - \sum_{m\neq k} \frac{\mu_m\mu_k^{-2}}{1-\mu_m\mu_k^{-1}} \tag{21b}$$

$$\frac{1}{\delta_k}\frac{\partial\delta_k}{\partial\nu_\ell} = -\frac{\mu_k}{1-\nu_\ell\mu_k} - \frac{\mu_k^{-1}}{1-\nu_\ell\mu_k^{-1}} \tag{21c}$$

$$\frac{1}{\delta_k}\frac{\partial\delta_k}{\partial\sigma^2} = \frac{1}{\sigma^2} . \tag{21d}$$

Equations (21a) - (21d) provide the quantities needed to evaluate $\partial\theta^{(2)}/\partial\theta^{(1)}$. Next recall that

$$\mu_k = e^{\gamma_k T} , \tag{22}$$

and

$$\frac{\partial\gamma_k}{\partial\mu_\ell} = \begin{cases} \frac{1}{T\mu_k} \; ; & \ell = k \\ 0 \; ; & \ell \neq k \end{cases} . \tag{23}$$

We can now evaluate $\partial\theta^{(3)}/\partial\theta^{(2)}$. Next note that

$$\delta_k = \frac{\gamma_1\lambda_k^{n-1} + \dots \gamma_n}{\prod_{m\neq k}(\lambda_k-\lambda_m)} , \tag{24}$$

and therefore

$$\log\delta_k = \log(\gamma_1\lambda_k^{n-1} + \dots + \gamma_n) - \sum_{m\neq k}\log(\lambda_k-\lambda_m) . \tag{25}$$

Differentiating (25) yields

$$\frac{1}{\delta_k}\frac{\partial\delta_k}{\partial\lambda_\ell} = \frac{1}{\lambda_k-\lambda_\ell} \quad ; \quad \ell \neq k \tag{26a}$$

$$\frac{1}{\delta_k}\frac{\partial\delta_k}{\partial\lambda_k} = \frac{(n-1)\gamma_1\lambda_k^{n-2} + \dots + \gamma_{n-1}}{\gamma_1\lambda_k^{n-1} + \dots + \gamma_n} - \sum_{m\neq k}\frac{1}{(\lambda_k-\lambda_m)} \tag{26b}$$

$$\frac{1}{\delta_k}\frac{\partial\delta_k}{\partial\gamma_\ell} = \frac{\lambda_k^{n-\ell}}{\gamma_1\lambda_k^{n-1} + \dots + \gamma_n} . \tag{26c}$$

Equations (26a) - (26c) make it possible to evaluate $\partial\theta^{(3)}/\partial\theta^{(4)}$ and thus $\partial\theta^{(4)}/\partial\theta^{(3)}$.

Next we note that

$$\prod_{m=1}^{n} (s-\lambda_m) = s^n + \alpha s^{n-1} + \ldots + \alpha_n . \tag{27}$$

Differentiating the equation above gives

$$-\sum_{i=1}^{n} \Big[\prod_{m\neq i}(s-\lambda_m)\Big]\frac{\partial\lambda_i}{\partial\alpha_\ell} = s^{n-\ell} , \tag{28}$$

and substituting $s = \lambda_k$ yields

$$\frac{\partial\lambda_k}{\partial\alpha_\ell} = - \frac{\lambda_k^{n-\ell}}{\prod_{m\neq k}(\lambda_k-\lambda_m)} . \tag{29}$$

This makes it possible to evaluate $\partial\theta^{(4)}/\partial\theta^{(5)}$ and hence $\partial\theta^{(5)}/\partial\theta^{(4)}$. Finally, recall that

$$\underline{\gamma} = \frac{1}{2} A^{-1}B \ \underline{\beta} . \tag{30}$$

Differentiation yields

$$\frac{\partial\underline{\gamma}}{\partial\alpha_\ell} = - \frac{1}{2} A^{-1} \frac{\partial A}{\partial\alpha_\ell} A^{-1}B \ \underline{\beta} = - A^{-1} \frac{\partial A}{\partial\alpha_\ell} \underline{\gamma} \tag{31a}$$

$$\frac{\partial\underline{\gamma}}{\partial\beta_\ell} = \frac{1}{2} A^{-1} \frac{\partial B}{\partial\beta_\ell} \underline{\beta} + \frac{1}{2} A^{-1}B \frac{\partial\underline{\beta}}{\partial\beta_\ell} = A^{-1} \frac{\partial B}{\partial\beta_\ell} \underline{\beta} , \tag{31b}$$

where

$$\Big\{\frac{\partial A}{\partial\alpha_\ell} \underline{\gamma}\Big\}_i = \begin{cases} (-1)^{n-\ell}\gamma_{2i-\ell} & ; \ 1 < 2i-\ell < n \\ 0 & ; \ \text{otherwise} \end{cases} \tag{31c}$$

$$\Big\{\frac{\partial B}{\partial\beta_\ell} \underline{\beta}\Big\}_i = \begin{cases} (-1)^{n-\ell}\beta_{2i-\ell} & ; \ 1 < 2i-\ell < n \\ 0 & ; \ \text{otherwise} \end{cases} . \tag{31d}$$

From equations (31a)-(31d) we can evaluate $\partial\theta^{(5)}/\partial\theta^{(6)}$, and thus $\partial\theta^{(6)}/\partial\theta^{(5)}$.

4. THE EXISTENCE OF AN OPTIMAL SAMPLING RATE

In this section we examine the behavior of the bound derived earlier, as a function of the sampling interval T. First, we note the following: A stationary process with a rational spectral density function has an infinite bandwidth, and an ideal reconstruction from the samples is impossible. However, there exists a sampling rate allowing a unique reconstruction of the process parameters from the ARMA parameters of the sampled process. This critical rate is determined by the requirement that all the discretized roots $\{e^{\lambda_m T}, 1 < m < n\}$ have phase angles in the range $[-\pi,\pi]$, i.e.

$$T < \min_{1<m<n} \left\{\frac{\pi}{|\mathrm{IM}(\lambda_m)|}\right\} , \tag{32}$$

where $\mathrm{IM}(\cdot)$ denotes the imaginary part of the complex argument. If all the $\{\lambda_m\}$ are real, the sampling interval T can be made arbitrarily large.

Let us now examine the case of a first-order rational spectrum, i.e.

$$S_x(s) = \frac{\beta^2}{(\alpha+s)(\alpha-s)} , \ \alpha > 0 . \tag{33}$$

The corresponding discrete-time spectrum is

$$S_y(z) = \frac{\sigma^2}{(1-\mu z)(1-\mu z^{-1})} , \tag{34}$$

where

$$\mu = e^{-\alpha T} \quad ; \quad \sigma^2 = \frac{\beta^2(1-e^{-2\alpha T})}{2\alpha} . \tag{35}$$

In this case we get

$$I_N\{\theta^{(1)}\} = N \cdot \begin{bmatrix} (1-e^{-2\alpha T})^{-1} & 0 \\ 0 & (2\sigma^4)^{-1} \end{bmatrix} \tag{36}$$

$$\frac{\partial\theta^{(6)}}{\partial\theta^{(1)}} = \begin{bmatrix} -Te^{-\alpha T} & 0 \\ \beta^2(1-2\alpha T)(1-e^{-2\alpha T})/2\alpha^2 & \beta(1-e^{-2\alpha T})/\alpha \end{bmatrix}^{-1} \qquad (37)$$

A convenient scalar measure of the magnitude of the estimation error is the so-called generalized variance, which is the determinant of the error covariance matrix. This is bounded from below by the determinant of the CRB [4, p. 195]. We define

$$d(\theta^{(6)}) \triangleq |CRB(\theta^{(6)})| = \frac{\left|\frac{\partial\theta^{(6)}}{\partial\theta^{(1)}}\right|^2}{\left|I_N(\theta^{(1)})\right|} . \qquad (38)$$

For the first-order case we get, using (36),(37)

$$d(\theta^{(6)}) = \frac{\alpha^2\beta^2}{2N^2} \cdot \frac{e^{2\alpha T}-1}{(\alpha T)^2} . \qquad (39)$$

Consider now the case in which the sampling interval T varies, while the total number of samples N remains fixed. The continuous-time parameters α and β are also assumed to be fixed. For both $T \to 0$ and $T \to \infty$, the bound on the generalized variance goes to infinity, as can be verified by using L'Hospital's rule. Hence there exists a global minimum, which was evaluated numerically to occur at $T \approx 0.8\alpha^{-1}$. The behavior of $d(\theta^{(6)})$ as a function of T is shown in Figure 1, where $\alpha=1$ and where $d(\theta^{(6)})$ was normalized by its minimum value.

The sampling interval $T \approx 0.8\alpha^{-1}$ is optimal in the sense of minimizing the best achievable generalized variance of the estimated continuous-time parameters. We conclude that for first-order rational spectra, there exists an optimal sampling rate for reconstructing the parameters of the continuous-time process.

Higher-order cases appear to be too complicated to obtain closed-form expressions. However, the formulas derived in Sections 2 and 3 can still be used to evaluate the bound $d(\theta^{(6)})$ for any given values of $\{\alpha_m, \beta_m\}$. Let us consider two further examples. The first one is that of the second order

spectrum

$$S_x(s) = \frac{(1+s)(1-s)}{(1+s+s^2)(1-s+s^2)} \ . \tag{40}$$

The bound, normalized with respect to its minimum value, is shown in Figure 2. Again we observe the existence of an optimal sampling interval, which is about 0.9. The curve is relatively flat over the range $0.5 < T < 1.5$, and very steep outside this range.

For the next example we chose the fourth-order spectrum

$$S_x(s) = \frac{(1+s+s^2+s^3)(1-s+s^2-s^3)}{(2+4s+5s^2+3s^3+s^4)(2-4s+5s^2-3s^3+s^4)} \ . \tag{41}$$

The normalized bound is shown in Figure 3. Here the optimal sampling rate is approximately 0.5, with a flat range of about $0.25 < T < 0.75$.

Finally, we illustrate the effect of the damping coefficient of the process on the behavior of generalized variance. We take the second-order spectrum

$$S_x(s) = \frac{(1+s)(1-s)}{(1+2\zeta s + s^2)(1-2\zeta s+s^2)} \ . \tag{42}$$

where ζ is the damping coefficient. Figures 4 and 5 show the normalized bound for $\zeta = 0.9$ and $\zeta = 0.1$ respectively. As we see, the flat region for $\zeta = 0.1$ is about twice as wide as the flat region for $\zeta = 0.9$. In other words, highly damped processes appear to be less sensitive to the choice of sampling rate than slightly damped process.

5. DISCUSSION

We have derived closed form expressions for the Crmaer-Rao lower bound on the covariance and the generalized variance of the estimated parameters of a continuous-time rational spectrum from measurements of a uniformly sampled realization of the given process. We explored the dependence of the CRB on the sampling interval and demonstrated the existence of an optimal sampling interval, in the sense of minimizing the CRB of the generalized variance for a fixed number of measurements.

Since the optimal sampling interval depends on the process parameters, it is reasonable to ask whether the above mentioned phenomenon can be used in practice. A common practical situation is one in which a continuous-time process with slowly time-varying spectrum is sampled, and where batches of data are processed in succession, so as to track the time variation of the spectrum. An adaptive sampling-rate adjustment procedure can be incorporated in such situations, as follows. Each time a batch is processed and the process parameters are identified, the method described in this paper can be used to compute the optimal sampling rate, which is then used as the sampling rate for the next batch. If the time variation of the process parameters is sufficiently slow, this will result in a nearly optimal sampling rate.

The same idea can be applied to an off-line processing of analog-recorded continuous-time signals. Here the sampling rate adjustment would be iterative rather than recursive, where at each iteration the same analog data is re-sampled.

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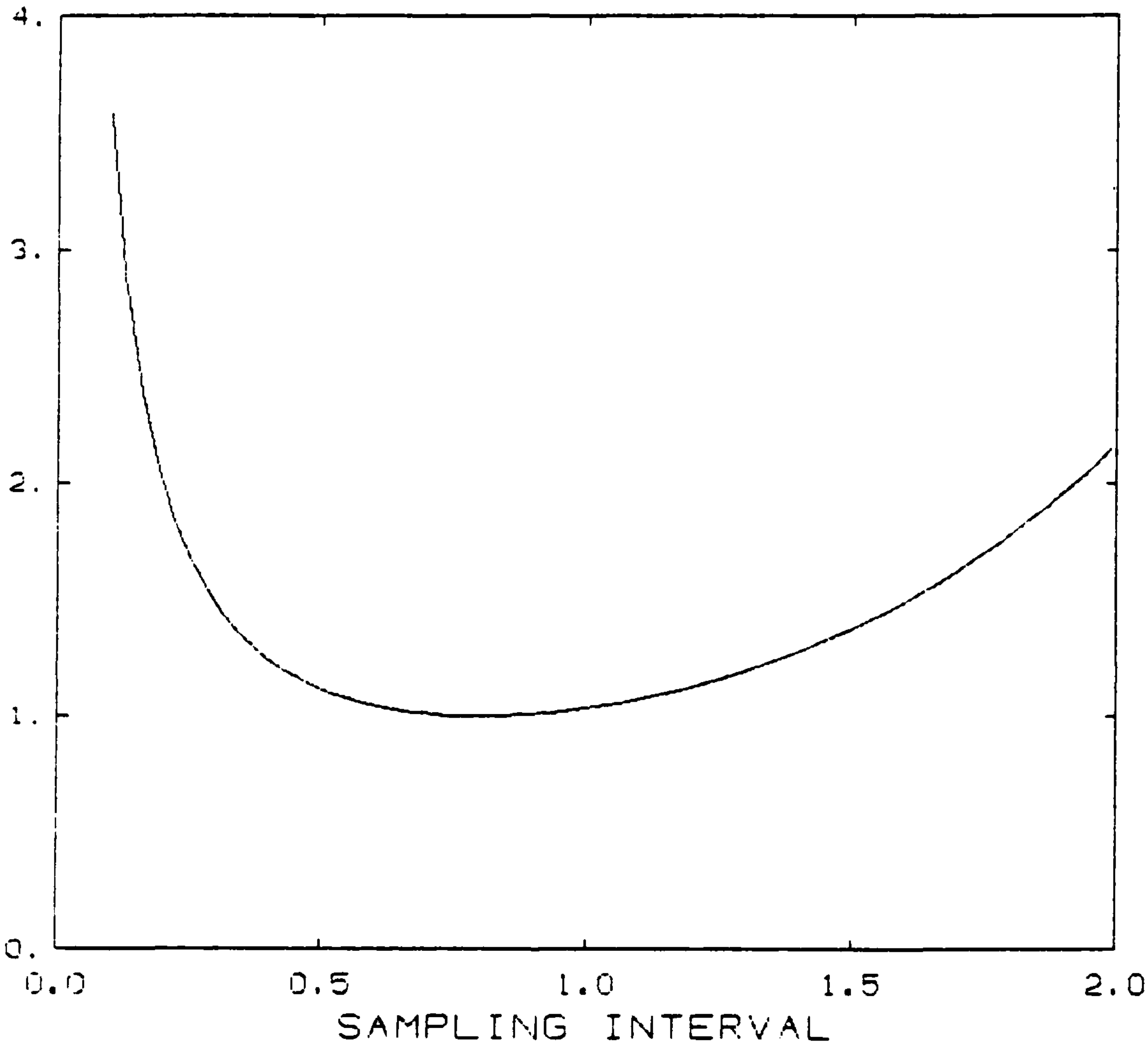

Figure 1. The Normalized Bound as a Function of the Sampling Interval: A First Order Process

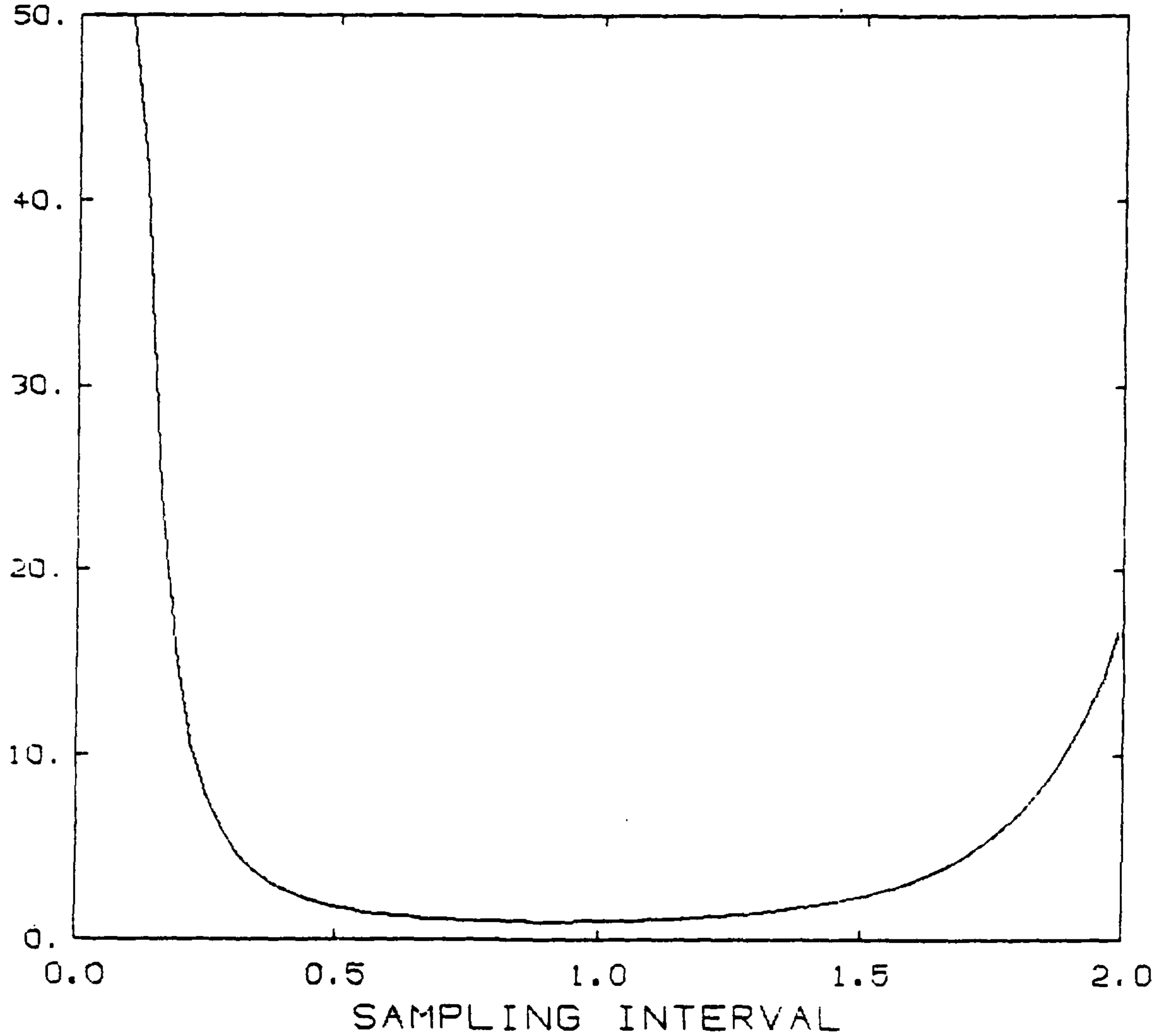

Figure 2. The Normalized Bound as a Function of the Sampling Interval: A Second Order Process

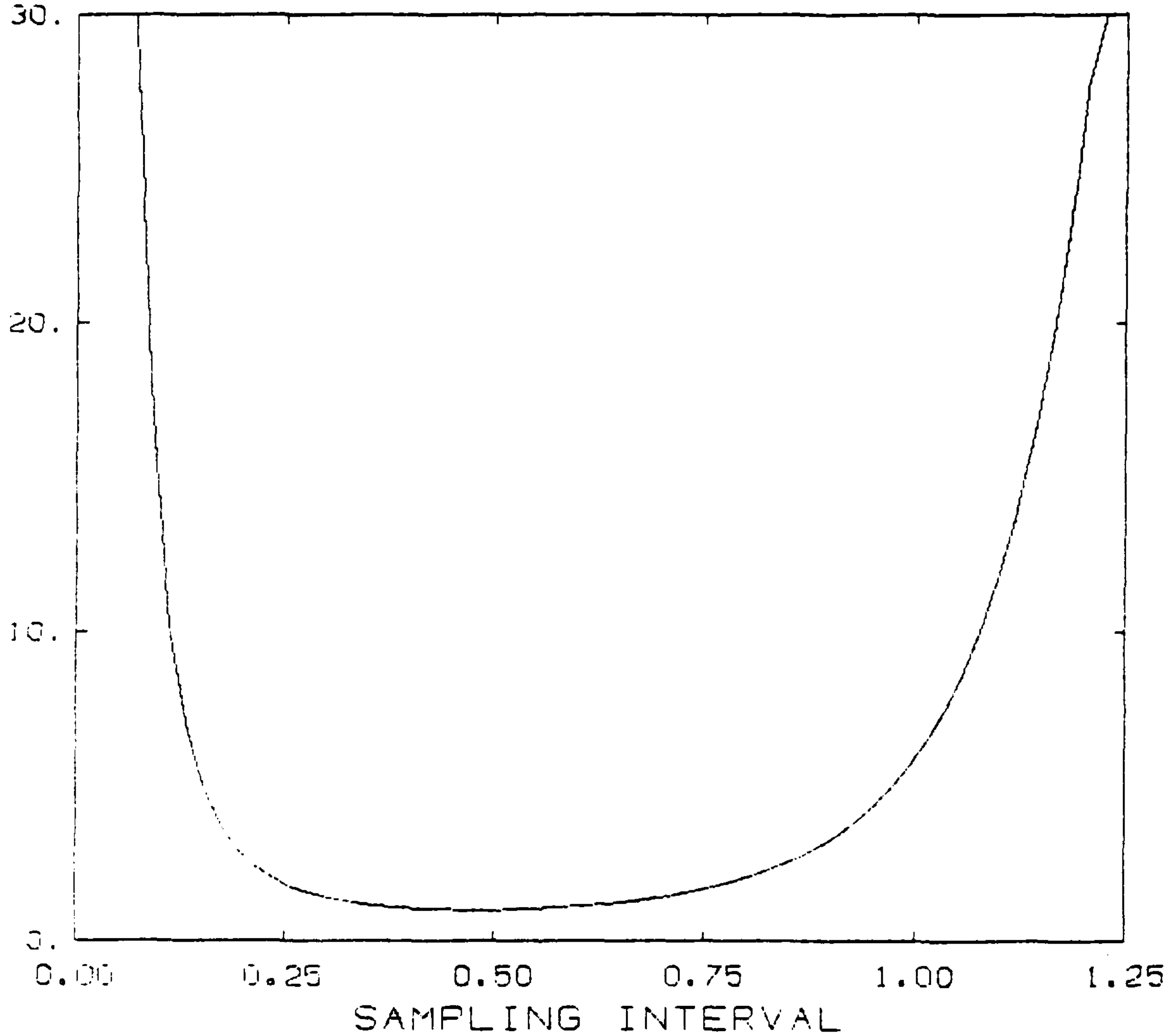

Figure 3. The Normalized Bound as a Function of the Sampling Interval: A Fourth Order Process

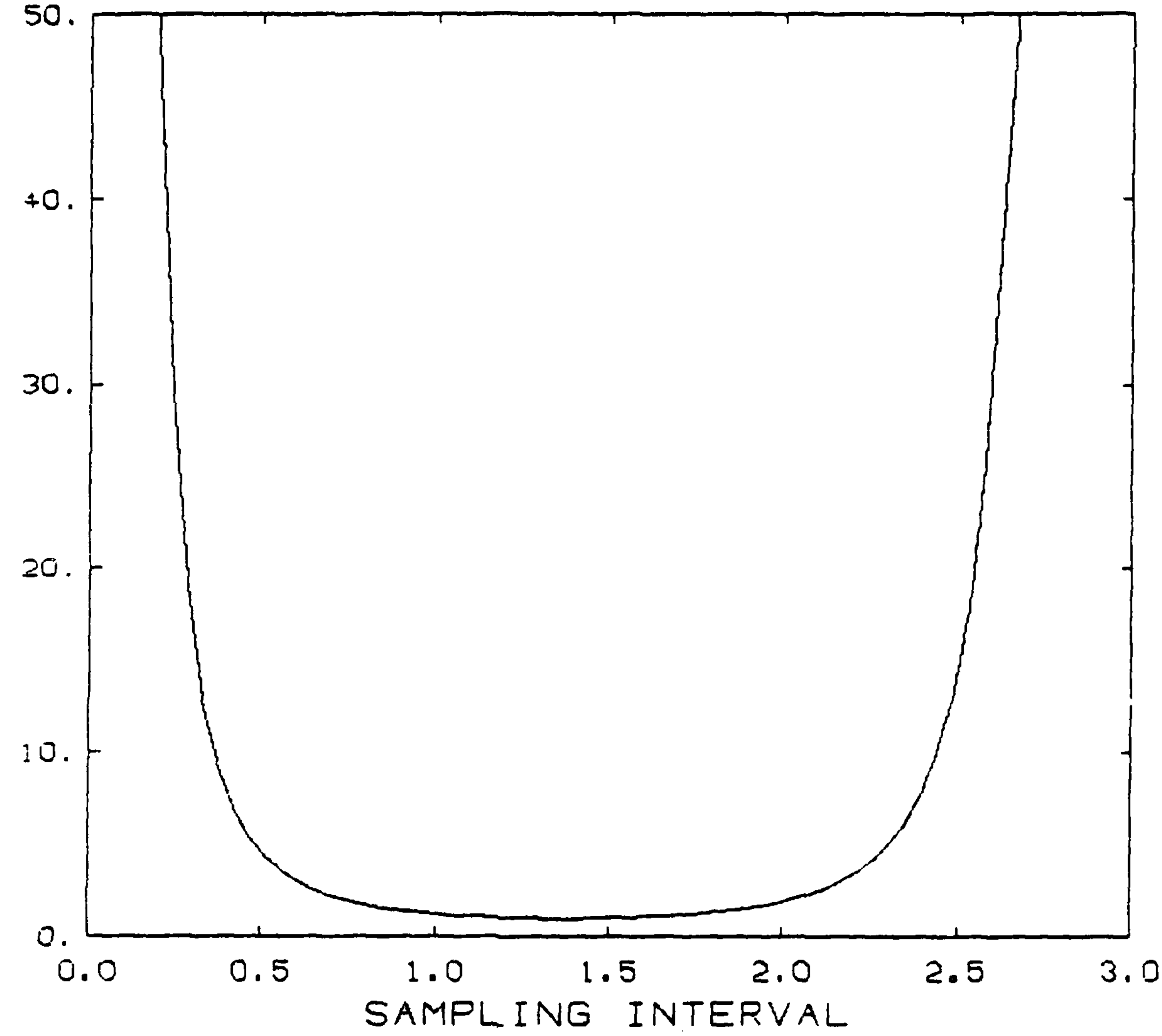

Figure 4. The Normalized Bound as a Function of the Sampling Interval: A Second Order Process with $\zeta = 0.9$

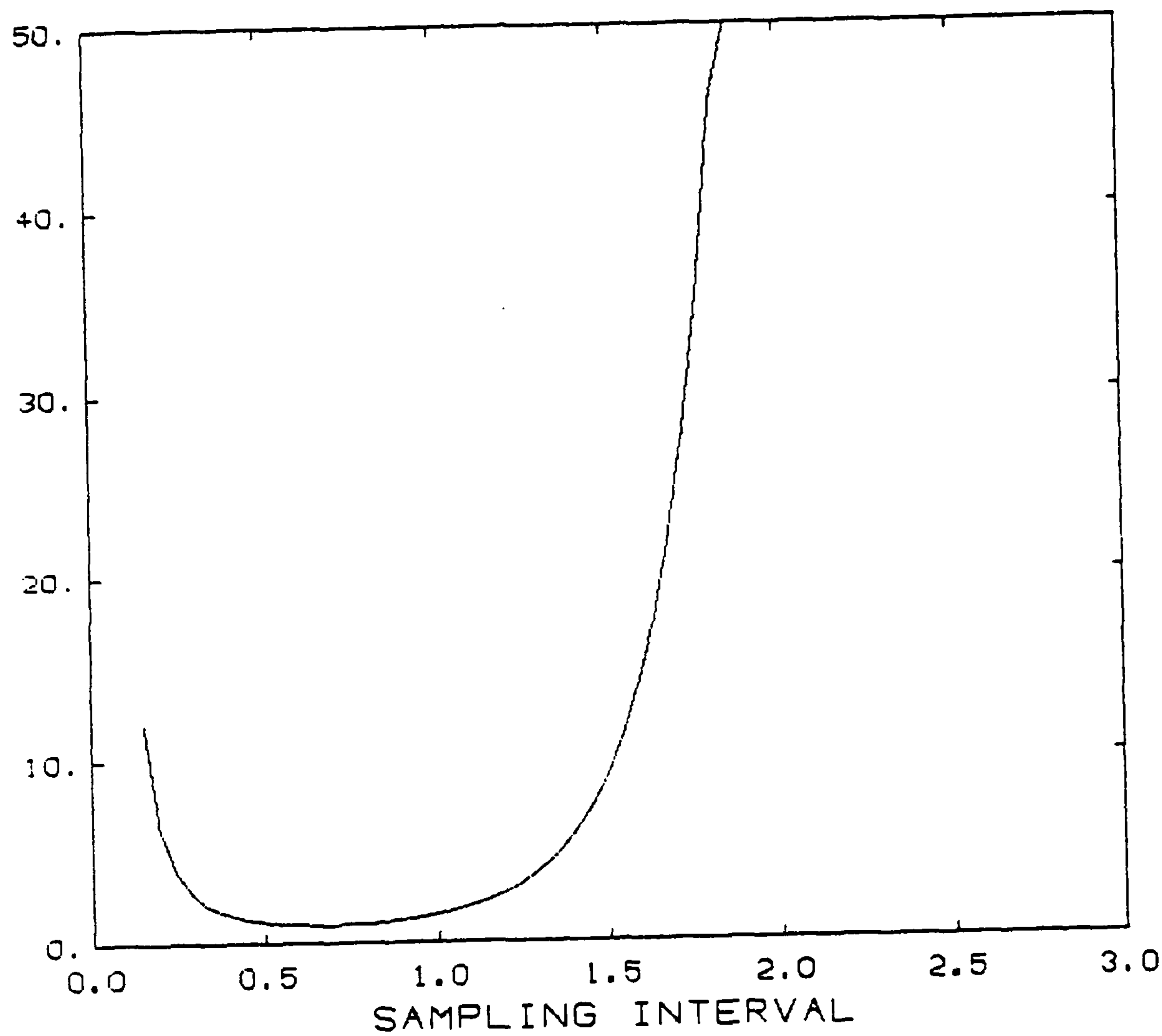

Figure 5. The Normalized Bound as a Function of the Sampling Interval: A Second Order Process with ζ = 0.1

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